Definability of Accelerated Relations in a Theory of Arrays and its Applications

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Context: Reachability analysis

$$\mathcal{S}_T \;=\; (\; \mathbf{v} \;,\; I(\mathbf{v}) \;,\; \tau(\mathbf{v}, \mathbf{v}') \;)$$

- **Ingredients**: transition system $\mathcal{S}_T$ and a safety property $P(\mathbf{v})$
- **Reachability analysis**: establish if it is possible to reach $\neg P(\mathbf{v})$

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⇒ T is Presburger arithmetic enriched with free function symbols

- satisfiability and validity with respect to structures having the standard structure of natural numbers as reduct
- $\mathbf{v}$ contains free unary function symbols ($\mathbf{a}$) and free constants ($\mathbf{c}$)

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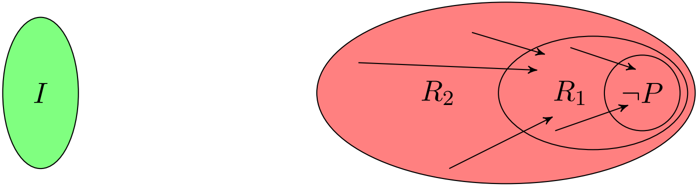

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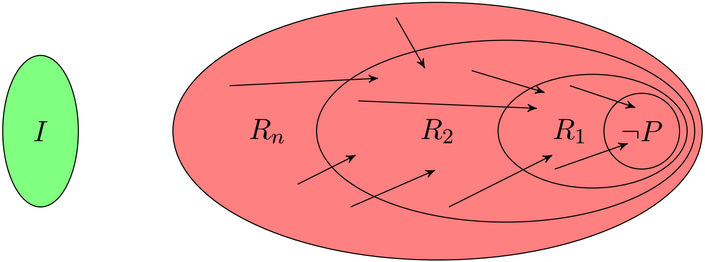

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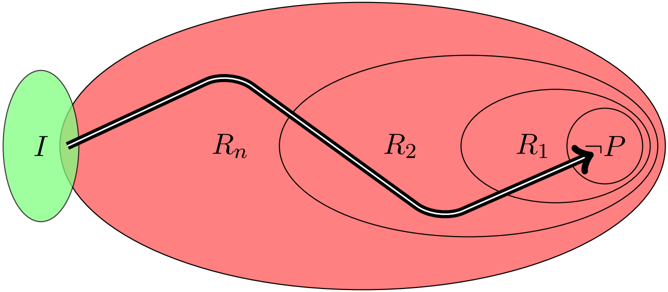

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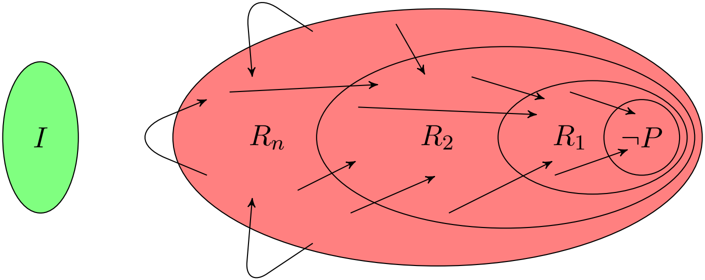
- ... until we find an intersection with the set of initial states...
- ... or a (global) fix-point.



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Reduce intersection and fix-point test to SMT problems:

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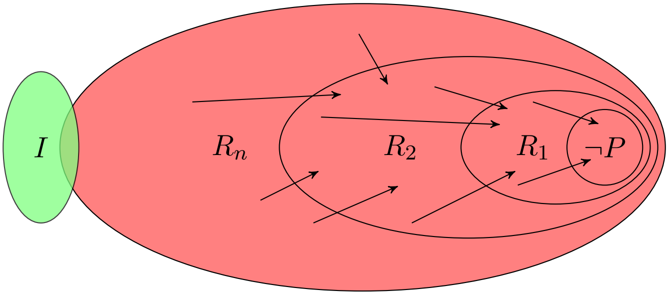

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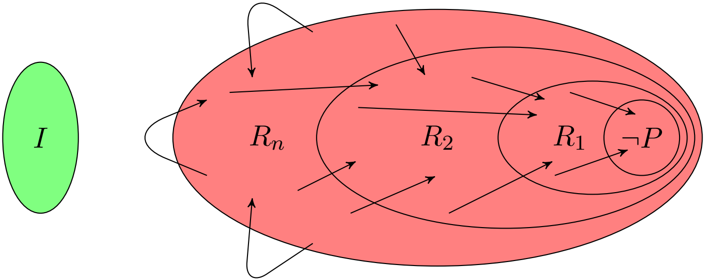
Reduce intersection and fix-point test to SMT problems:

- Intersection test: is $I \wedge R_n$ T-satisfiable?
- Fix-point test: is $R_{n+1} \rightarrow R_n$ T-valid?
- ...or dually: is $R_{n+1} \wedge \neg R_n$ T-unsatisfiable?



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$\Rightarrow$ Acceleration can help in limiting divergence!

```
        procedure Find( int e ) {
l_I         i = 0;
l_L         while ( i < L ∧ a[i] ≠ e ) {
               i = i + 1;
            }
l_F         assert ( ∀x.(0 ≤ x < i) → a[x] ≠ e );
        }
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[1] Assume we exit the loop because we reach the end of the array.

Acceleration

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$$\tau_1 := pc = l_L \quad \wedge \quad \underbrace{\mathtt{i} < \mathtt{L} \wedge \mathtt{a}[\mathtt{i}] \neq \mathtt{e}}_{\text{guard}} \quad \wedge \quad \underbrace{\mathtt{i}' = \mathtt{i} + 1}_{\text{update}}$$

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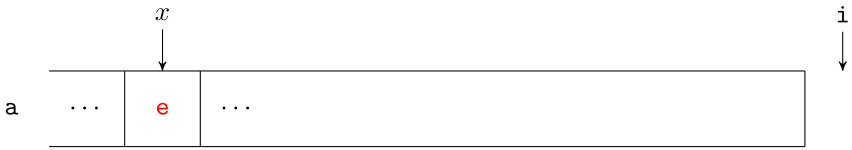

$$\exists x. 0 \leq x \wedge x < i \wedge a[x] = e \wedge i \geq L$$

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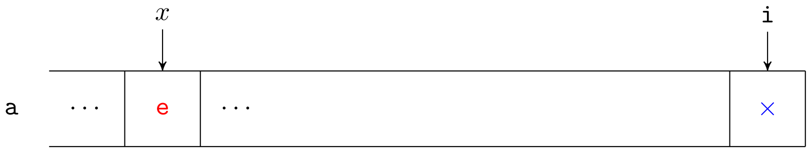

$\exists x. 0 \leq x \land x < i+1 \land a[x] = e \land i+1 = L \land$
$a[i] \neq e$

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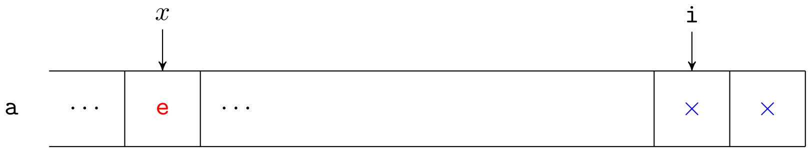

$$\exists x. 0 \leq x \land x < i + 2 \land a[x] = e \land i + 2 = L \land$$
$$a[i] \neq e \land a[i+1] \neq e$$

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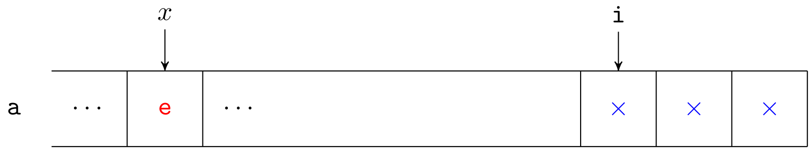

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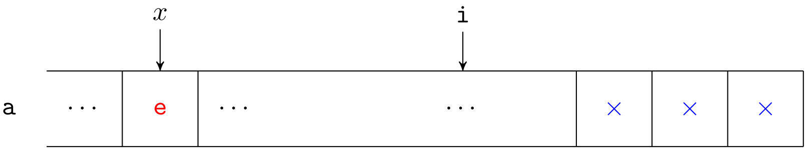

$$\exists x. 0 \leq x \wedge x < i + n \wedge a[x] = e \wedge i + n = L \wedge$$

$$\bigwedge_{k=0}^{n-1} a[i+k] \neq e$$

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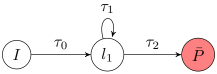
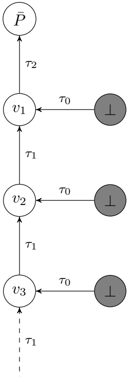
Acceleration
Preventing divergence

Find control-flow graph:



Precise backward reachability



With accelerated transitions

(desired behavior)

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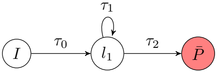

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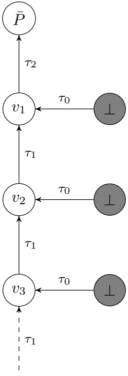

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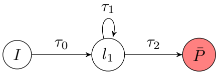

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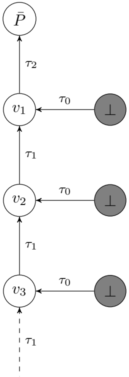

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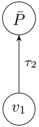

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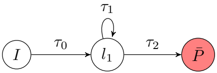

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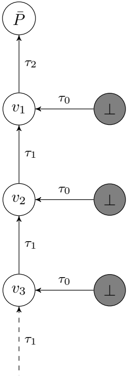

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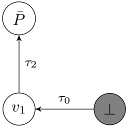

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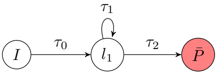

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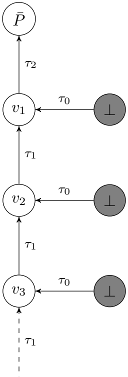

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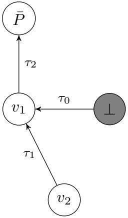

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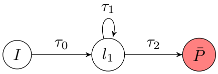

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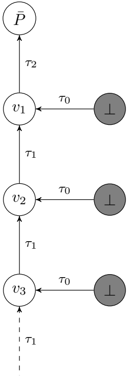

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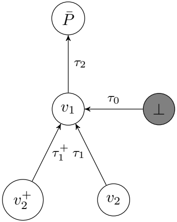

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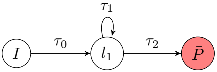

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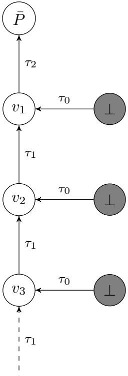

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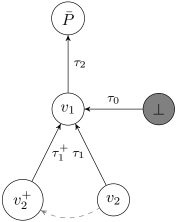

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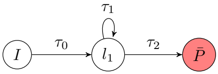

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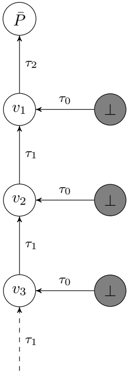

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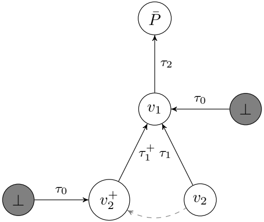

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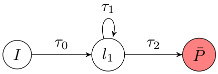

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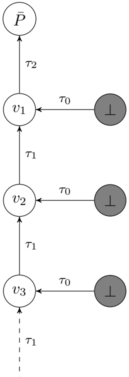

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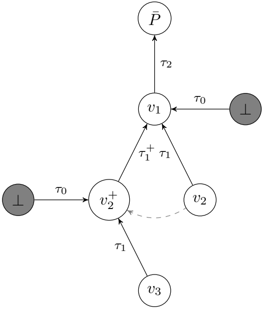

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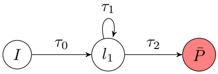

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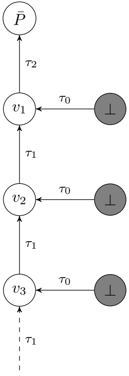

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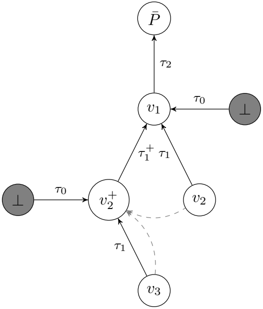

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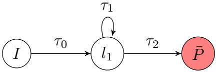

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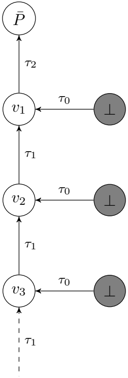

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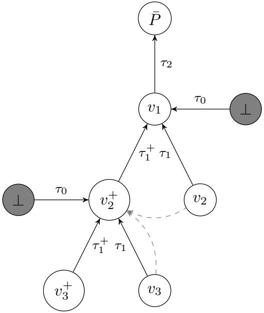

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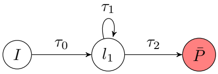
Find control-flow graph: 

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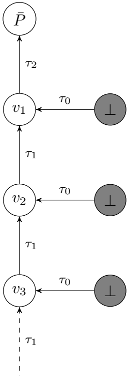

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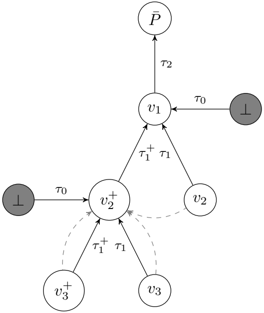

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- Template-based solution
  - ✔ High degree of automation
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In practice:

- Template-based solution
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  - ✔ Computationally cheap
- Combination with abstraction-based frameworks

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$$\tau_1^+ := \exists y. \begin{pmatrix} y > 0 \wedge pc = l_L \ \wedge \\ \forall j.(\ \mathtt{i} \leq j < \mathtt{i} + y \quad \rightarrow \quad j < \mathtt{L} \wedge \mathtt{a}[j] \neq \mathtt{e}\ ) \\ \mathtt{i}' = \mathtt{i} + y \end{pmatrix}$$

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Definition (Iterators)

A tuple of m-ary terms $\mathbf{u}(\underline{x})$ is said to be an *iterator* iff there exists an m-tuple of $m+1$-ary terms $\mathbf{u}^*(\underline{x}, y)$ such that for any natural number n it happens that the formula

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$\mathbf{u}(x) := x + 1$

The formal framework

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$\mathbf{u}(x) := x + 1$

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Definition (Selectors)

Given an iterator $\mathbf{u}(\underline{x})$, an m-ary term $\kappa(x_1, \ldots, x_m)$ is a *selector* for $\mathbf{u}(\underline{x})$ iff there is an $m+1$-ary term $\iota(x_1, \ldots, x_m, y)$ yielding the validity of the formula

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- Can a cell z be reached in m iterations?
- The number $\iota(\underline{x}, z)$ gives "the only possible candidate" y number of iterations
- $z = \kappa(\mathbf{u}^*(\underline{x}, y))$ checks if the candidate y is correct

$$\texttt{while}\ (\ \texttt{true}\ )\ \{\ a[i] = 0; i = i + 2;\ \}$$

The formal framework
Example

$$\texttt{while (true) } \{\ a[i] = 0; i = i + 2;\ \}$$

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The formal framework

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Definition (Local ground assignment)

A *local ground assignment* is a ground assignment of the form

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\begin{aligned}
& pc = l \ \wedge \ \phi_L(\mathbf{a}, \mathbf{c}) \ \wedge \ pc' = l \ \wedge \\
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The formal framework

Contribution

Theorem

If τ is a local ground assignment, then τ^+ is a Σ^0_2-assignment.

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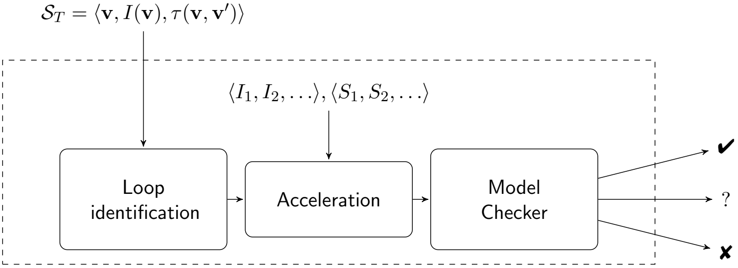
- The proof of the theorem shows the "template" for τ^+
- The template is parametric with respect to
  - iterators
  - selectors

Tool architecture



Acceleration for arrays

Practical issue - classification of formulas

Different kind of formulas[2] representing the (backward reachable) state-space:

- *ground* – formulas of the kind $\phi(\mathbf{v})$

[2] In all the formulas we admit the term $a(t)$ only if t is a variable or a constant.

Acceleration for arrays

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Σ^0_2-formulas might not fall in any known decidable fragment [BMS06, GdM09]

[2] In all the formulas we admit the term $a(t)$ only if t is a variable or a constant.

Transition formulas can be:

- *ground* assignment – transitions of the kind $\tau(\mathbf{v}, \mathbf{v}')$
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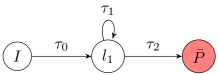
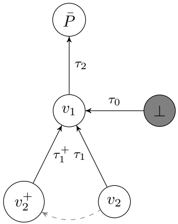
- Preimages with respect to a Σ_2^0-assignment are Σ_2^0-formulas
- This prevents the practical application of the theoretical result!
- Solution: over-approximate problematic Σ_2^0-formulas with their *monotonic abstraction* [AGP+12]

Acceleration for arrays

Example





Acceleration for arrays

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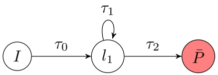

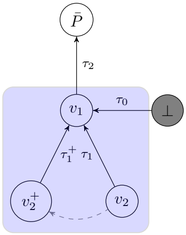

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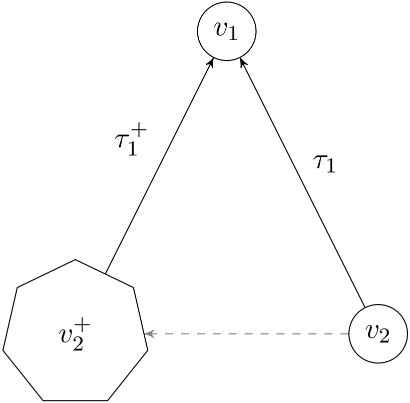

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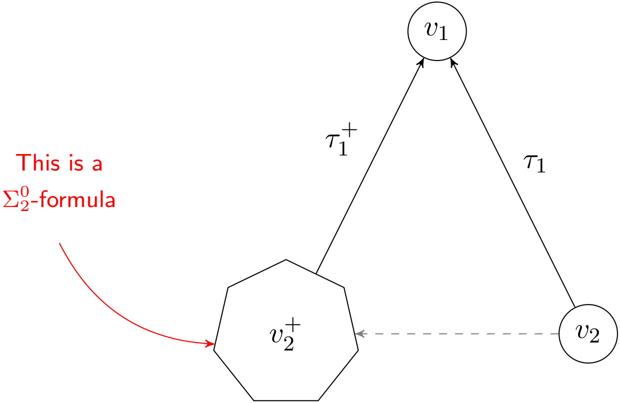

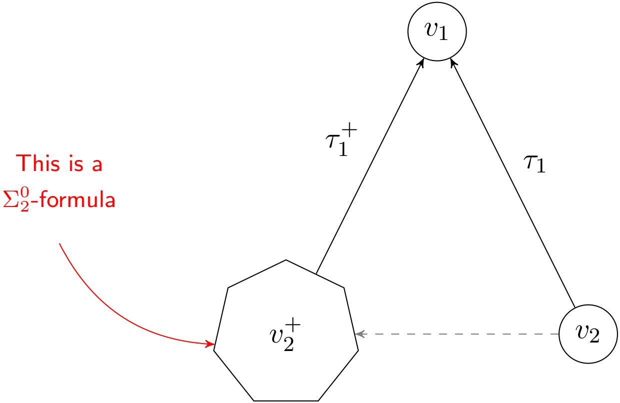

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Acceleration for arrays

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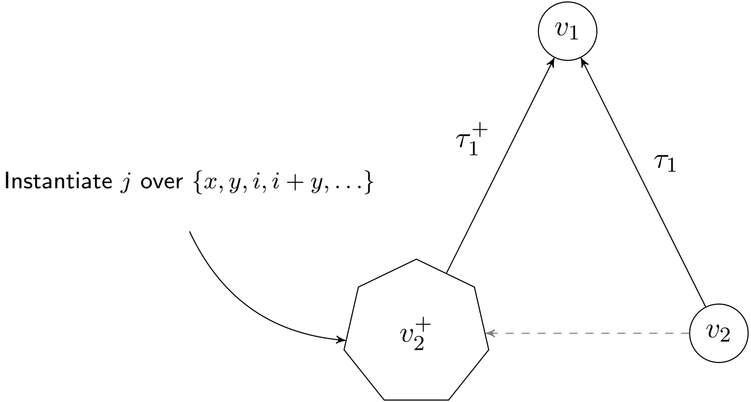

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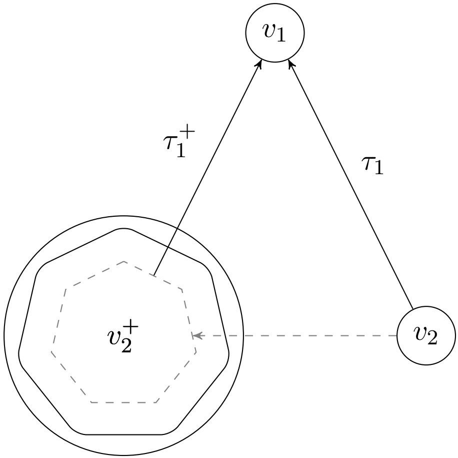

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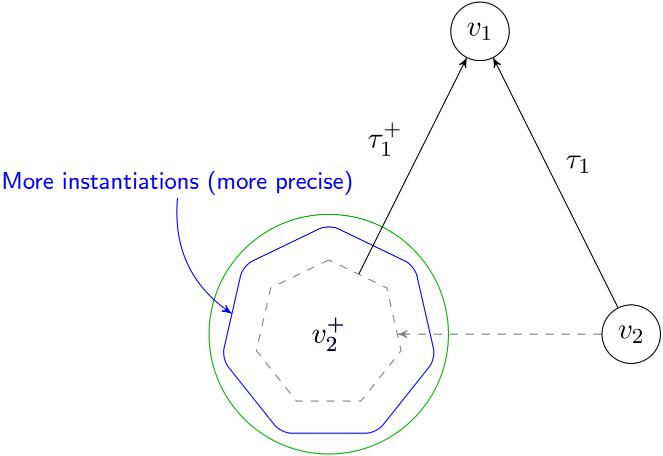

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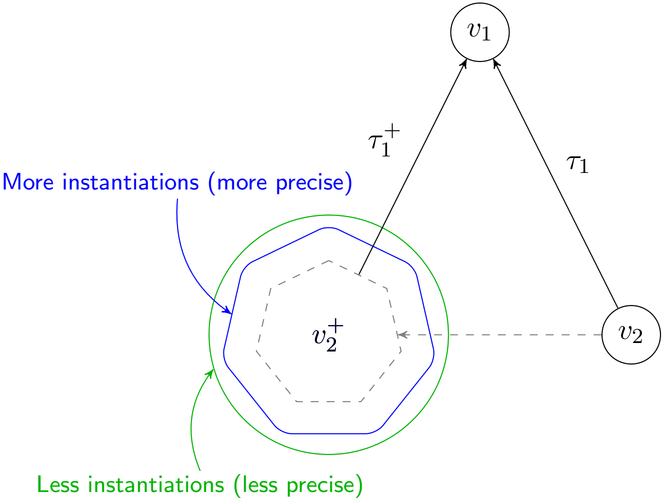

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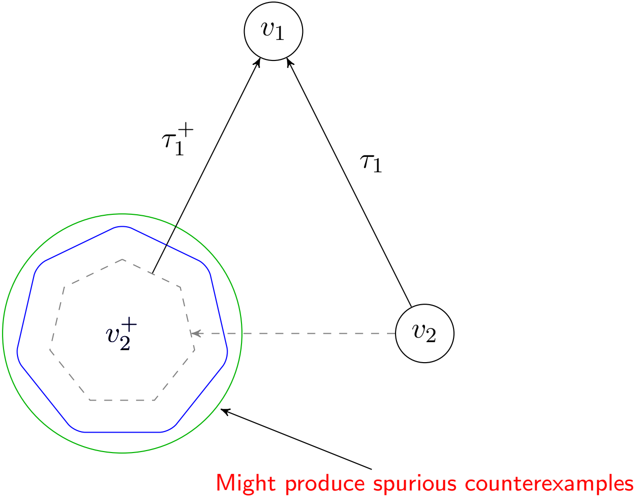
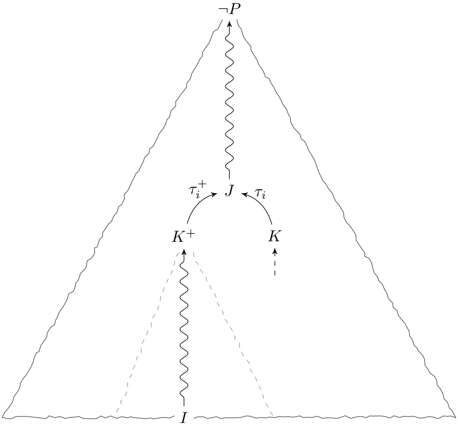


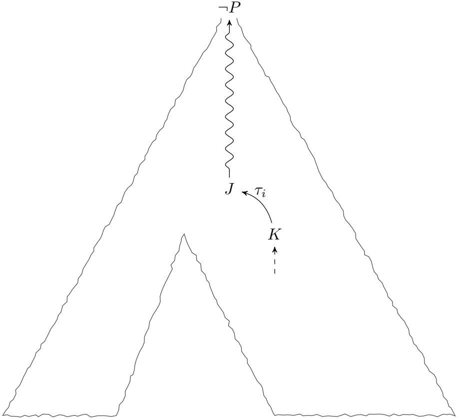
Acceleration for arrays

Ad-hoc refinement for monotonic abstraction



Acceleration for arrays

Ad-hoc refinement for monotonic abstraction



Acceleration for arrays
Experiments

- Implemented in the MCMT model checker

Acceleration for arrays
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- Tested on 55 challenging benchmarks on arrays
  - initializing
  - searching
  - sorting
  - etc.

Acceleration for arrays

Experiments

```
function allDiff ( int a[N] ) :
r = true;
for (i = 1; i < N ∧ r; i++)
    for (j = i-1; j ≥ 0 ∧ r; j--)
        if (a[i] = a[j]) r = false;
assert (r → (∀x, y(0 ≤ x < y < N) → (a[x] ≠ a[y])))
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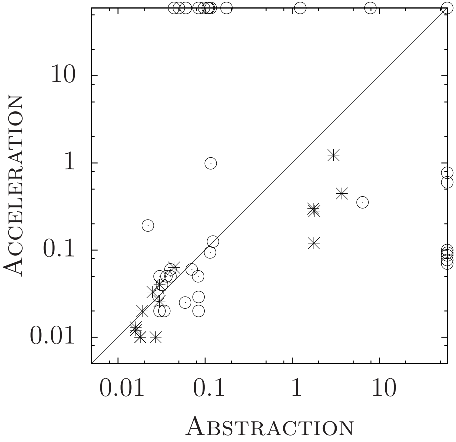
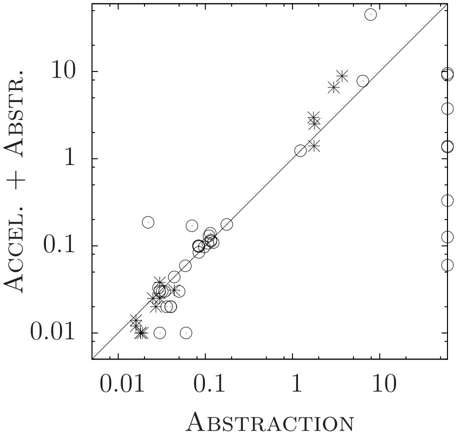
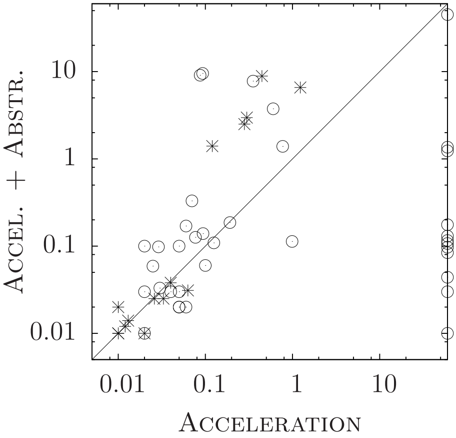


MCMT running time



MCMT running time



MCMT running time

Conclusion

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Thank you! Questions?

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