Specification and Verification of Linear Dynamical Systems: Advances and Challenges

Joël Ouaknine

Department of Computer Science, Oxford University

(Joint work with James Worrell)

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Nancy, France

Specification and Verification

$$M \vDash \varphi$$

Specification and Verification

$$M \models \varphi$$

M: linear dynamical systems

Specification and Verification

$$M \models \varphi$$

M: linear dynamical systems (discrete / continuous)

Specification and Verification

$$M \models \varphi$$

M: linear dynamical systems (discrete / continuous)

φ: ???

Termination of Simple Linear Programs

$\mathbf{x} := \mathbf{a};$
while $\mathbf{u} \cdot \mathbf{x} \neq 0$ do
$\quad \mathbf{x} := \mathbf{M} \cdot \mathbf{x};$

Termination of Simple Linear Programs

$\mathbf{x} := \mathbf{a};$
while $\mathbf{u} \cdot \mathbf{x} \neq 0$ do
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Termination Problem

Instance: $\langle\ \mathbf{a};\ \mathbf{u};\ \mathbf{M}\ \rangle$
Question: Does this program terminate?

Termination of Simple Linear Programs

Much work on this and related problems in the literature over the last three decades:

- Manna, Pnueli, Kannan, Lipton, Sagiv, Podelski, Rybalchenko, Cook, Dershowitz, Tiwari, Braverman, Kovács, Ben-Amram, Genaim, . . .
- Approaches include:
  - linear ranking functions
  - size-change termination methods
  - spectral techniques
  - . . .
- Tools include:

TERMINATOR
proof tools for termination and liveness

ACL2

Reachability/Invariance/Approximation in Markov Chains

M: Markov chain over states $s_1, \ldots, s_k$

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Markov Chain Problem

Instance: $\langle$ stochastic matrix **M**; $r \in (0,1]$ $\rangle$

Question: Does $\exists T$ s.t. $\forall n \geq T$, $(1,0,\ldots,0) \cdot \mathbf{M}^n \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \geq r$?

Positivity of Linear Recurrence Sequences

$u_0 = 1,\ u_1 = 1$
$u_{n+2} = u_{n+1} + u_n$

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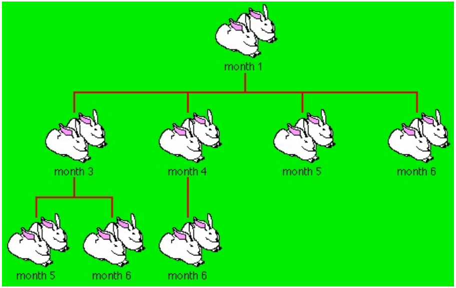

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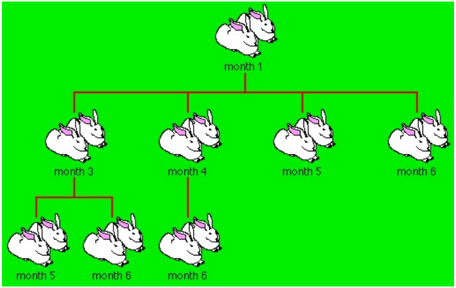

- 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, . . .

Positivity of Linear Recurrence Sequences

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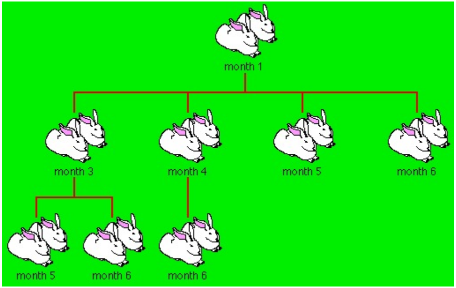

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$$u_0 = 1,\ u_1 = 1,\ u_2 = 2,\ u_3 = 3,\ u_4 = 5$$
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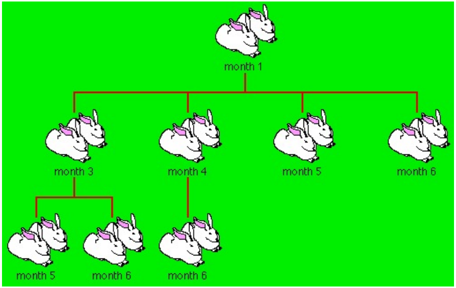

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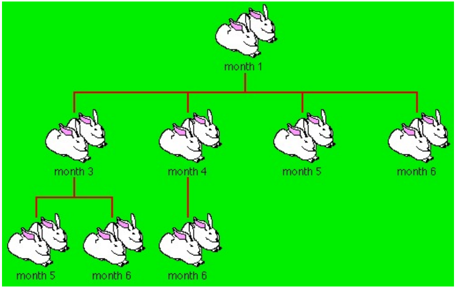

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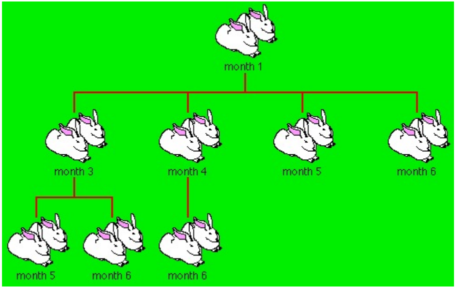

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Positivity Problem

Instance: A linear recurrence sequence $\langle u_n \rangle$
Question: Is it the case that $\forall n,\ u_n \geq 0$?

Sample Decision Problems

Termination Problem for Simple Linear Programs

Instance: $\langle\ \mathbf{a};\ \mathbf{u};\ \mathbf{M}\ \rangle$ over $\mathbb{Z}$

Question: Does this program terminate?

```
x := a;
while  u · x ≠ 0  do
    x := M · x;
```

Markov Chain Problem

Instance: A stochastic matrix $\mathbf{M}$ over $\mathbb{Q}$

Question: Does $\exists T$ s.t. $\forall n \geq T$, $(1, 0, \ldots, 0) \cdot \mathbf{M}^n \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \geq \frac{1}{2}$?

Positivity Problem for Linear Recurrence Sequences

Instance: A linear recurrence sequence $\langle u_n \rangle$ over $\mathbb{Z}$ or $\mathbb{Q}$

Question: Is it the case that $\forall n$, $u_n \geq 0$?

Linear Recurrence Sequences

Definition

A **linear recurrence sequence** is a sequence $\langle u_0, u_1, u_2, \ldots \rangle$ of real numbers such that there exist k and constants $a_1, \ldots, a_k$, such that

$$\forall n \geq 0, \quad u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \ldots + a_k u_n \, .$$

- k is the **order** of the sequence

Decision Problems for Linear Recurrence Sequences

- Let $\langle u_n \rangle$ be a linear recurrence sequence

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Does $\exists T$ such that, $\forall n \geq T,\ u_n \geq 0$?

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- *Effective* means T must also be provided.

Related Work and Applications

- Theoretical biology
  - Analysis of L-systems
  - Population dynamics
- Software verification
  - Termination of linear programs
- Probabilistic model checking
  - Reachability, invariance, and approximation in Markov chains
  - Stochastic logics
- Quantum computing
  - Threshold problems for quantum automata
- Economics
- Combinatorics
- Discrete linear dynamical systems
- Statistical physics
- ...

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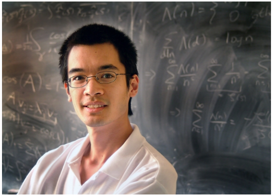

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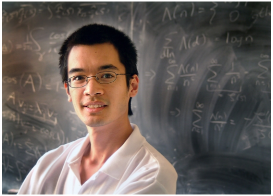

". . . a mathematical embarrassment . . ."

Richard Lipton

The Skolem-Mahler-Lech Theorem

Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semi-linear:

$$\{n : u_n = 0\} = F \cup A_1 \cup \ldots \cup A_\ell$$

where F is finite and each A_i is a full arithmetic progression.

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Theorem (Berstel and Mignotte 1976)

In Skolem-Mahler-Lech, the infinite part (arithmetic progressions $A_1, \ldots, A_\ell$) is fully effective.

The Skolem Problem at Low Orders

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Does $\exists n$ such that $u_n = 0$?

Let u_n be a linear recurrence sequence of fixed order

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For orders 1 and 2, Skolem is decidable.

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Theorem (Mignotte, Shorey, Tijdeman 1984; Vereshchagin 1985)

For orders 3 and 4, Skolem is decidable.

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Critical ingredient is Baker's theorem for linear forms in logarithms, which earned Baker the Fields Medal in 1970.

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For orders 3 and 4, Skolem is decidable.

Decidability for order 5 was announced in 2005 by four Finnish mathematicians in a technical report (as yet unpublished). Their proof appears to have a serious gap.

The Positivity and Ultimate Positivity Problems

- Positivity and Ultimate Positivity open since at least 1970s

"In our estimation, these will be very difficult problems."

Matti Soittola

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Theorem (folklore)

Decidability of Positivity $\Rightarrow$ decidability of Skolem.

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Theorem (Burke, Webb 1981)

For order 2, Ultimate Positivity is decidable.

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Theorem (Halava, Harju, Hirvensalo 2006)

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Theorem (Laohakosol and Tangsupphathawat 2009)

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In *Colloquium Mathematicum* 128:1 (2012), Tangsupphathawat, Punnim, and Laohakosol claimed decidability of Positivity and Ultimate Positivity for order 4 (and noted being stuck for order 5). Unfortunately, their proof contains a major error.

Our Main Results (I)

Theorem

- *Positivity is decidable for order 5 or less.*

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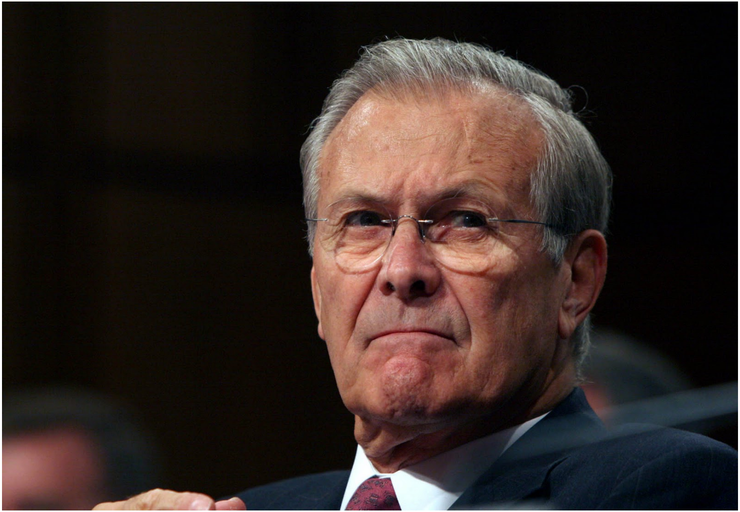
- *In the general case, complexity is in PSPACE and co$\exists\mathbb{R}$-hard.*

Known Unknowns



"There are things that we know we don't know..."

Donald Rumsfeld

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Moreover, $\frac{1}{\sqrt{5}}$ is the best possible constant that will work for all real numbers x.

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Almost nothing else is known about any specific irrational number!

Hardness

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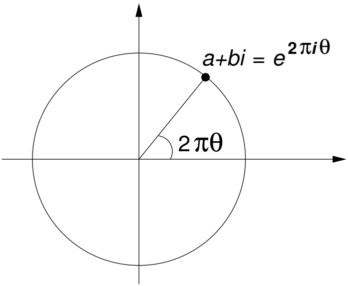

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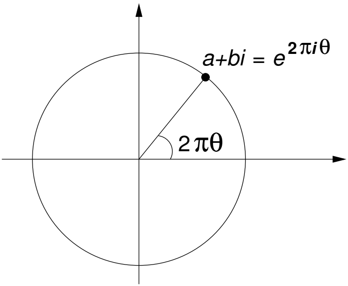

- $\mathcal{T}$ is a countable set of transcendental numbers

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- Several additional results hold (notably relating to the computability of *inhomogeneous* Diophantine approximation constants), and likewise for Positivity . . .

Main Tools and Techniques

- Algebraic and analytic number theory, Diophantine geometry
  - p-adic techniques
  - Baker's theorem on linear forms in logarithms
  - Kronecker's theorem on simultaneous Diophantine approximation
  - Masser's results on multiplicative relationships among algebraic numbers
  - Schmidt's Subspace theorem and Schlickewei's p-adic extension
  - Sums of S-units techniques
  - Gelfond-Schneider theorem
  - Other Diophantine geometry and approximation techniques
- Real algebraic geometry
- Decidability and fined-grained complexity of first-order theory of the reals (Renegar)

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- Decidability?
- Complexity?
- Synthesis problems: e.g., can we *compute* largest A such that program terminates for all $\mathbf{x} \in A$?

Discrete Linear Dynamical Systems

Definition

A **discrete linear dynamical systems** consists of a linear transformation $\mathbf{M}$ on a finite-dimensional vector space V.

- Typically $V = \mathbb{R}^n$ or $\mathbb{Q}^n$

Decision Problems for Linear Dynamical Systems

Definition

Given a vector $\mathbf{v} \in V$, the **orbit** of $\mathbf{v}$ under $\mathbf{M}$ is

$$\mathcal{O}_{\mathbf{M}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{M}\mathbf{v}, \mathbf{M}^2\mathbf{v}, \mathbf{M}^3\mathbf{v}, \ldots \rangle .$$

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- $\langle b_0, b_1, b_2, \ldots, b_j, \ldots \rangle$ is an LRS of order n (here $n = 4$)

Linear Dynamical Systems: Specification and Verification

- A fresh look at an old area
- Lots of challenging problems
- Lots of interesting maths
- Many connections to variety of other fields