Logically Constrained Term Rewriting Systems

Cynthia Kop and Naoki Nishida

Sept 20, 2013

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Frontiers of Combining Systems

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Term Rewriting

and

SMT

Term Rewriting: Traditional Example

$$\begin{aligned} \mathsf{sum}(0) &\to 0 \\ \mathsf{sum}(\mathsf{s}(x)) &\to \mathsf{plus}(\mathsf{s}(x), \mathsf{sum}(x)) \end{aligned}$$

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\begin{array}{rcl}
\mathsf{sum}(0) & \to & 0 \\
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Term Rewriting: Less Traditional Example

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Integers: $0, \mathsf{s}(0), \mathsf{s}(\mathsf{s}(0)), \ldots, \mathsf{p}(0), \mathsf{p}(\mathsf{p}(0)), \ldots$

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$$
\begin{aligned}
\mathsf{geq}(x, y) &\to \mathsf{geqhelp}(x, y, 0, 0) \\
\mathsf{geqhelp}(\mathsf{s}(x), y, z, a) &\to \mathsf{geqhelp}(x, y, \mathsf{s}(z), a) \\
\mathsf{geqhelp}(\mathsf{p}(x), y, z, a) &\to \mathsf{geqhelp}(x, y, z, \mathsf{s}(a)) \\
\mathsf{geqhelp}(0, \mathsf{s}(x), y, z) &\to \mathsf{geqhelp}(0, x, y, \mathsf{s}(z)) \\
\mathsf{geqhelp}(0, \mathsf{p}(x), y, z) &\to \mathsf{geqhelp}(0, x, \mathsf{s}(y), z) \\
\mathsf{geqhelp}(0, 0, \mathsf{s}(x), \mathsf{s}(y)) &\to \mathsf{geqhelp}(0, 0, x, y) \\
\mathsf{geqhelp}(0, 0, x, 0) &\to \mathsf{true} \\
\mathsf{geqhelp}(0, 0, 0, \mathsf{s}(x)) &\to \mathsf{false}
\end{aligned}
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Term Rewriting with Constraints

$$\begin{array}{lcll} \mathsf{sum}(x) & \to & 0 & [x \leq 0] \\ \mathsf{sum}(x) & \to & \mathsf{plus}(x, \mathsf{sum}(\mathsf{plus}(x, -1)) & [x > 0] \end{array}$$

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Term Rewriting with Constraints and Internal Calculation

$$\begin{array}{lcll} \mathsf{sum}(x) & \to & 0 & [x \leq 0] \\ \mathsf{sum}(x) & \to & x + \mathsf{sum}(x-1) & [x > 0] \end{array}$$

Modelling Imperative Programs

```
int sum(int x) {
  int i = 0, z = 0;
  for (i = 0; i <= x; i++)
    z += i;
  return z;
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\mathsf{sum}(x) & \to & \mathsf{u}(x,0,0) & \\
\mathsf{u}(x,i,z) & \to & \mathsf{u}(x,i+1,z+i) & [i \leq x] \\
\mathsf{u}(x,i,z) & \to & z & [\neg(i \leq x)]
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Modelling Imperative Programs

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int fac(x) {
  if (x <= 0) return 1;
  return x * fac(x-1);
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\mathsf{sum}(x) & \to & \mathsf{u}(x,0,0) & \\
\mathsf{u}(x,i,z) & \to & \mathsf{u}(x,i+1,z+\mathsf{fac}(i)) & [i \leq x] \\
\mathsf{u}(x,i,z) & \to & z & [\neg(i \leq x)] \\
\mathsf{fac}(x) & \to & x * \mathit{fac}(x-1) & [\neg(x \leq 0)] \\
\mathsf{fac}(x) & \to & 1 & [x \leq 0]
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Definitions of Rewriting with Integers and/or Constraints

- integer TRSs
- $\mathbb{Z}$-TRSs
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Integer TRSs: Idea

- infinite signature: add all integers, $\top, \bot$ and operators $+, -, *, /, \%, >, \geq, <, \leq, =, \neq, \wedge, \Rightarrow$
- infinite rules: add eg rules
  $1 + 1 \to 2$
  $6 \geq 9 \to \text{false}$
  $\text{true} \wedge \text{true} \to \text{true}$

Integer TRSs: Example

$$\begin{aligned} \mathsf{sum}(x) &\to \mathsf{sumhelp}(0 \geq x, x) \\ \mathsf{sumhelp}(\mathsf{true}, x) &\to 0 \\ \mathsf{sumhelp}(\mathsf{false}, x) &\to x + \mathsf{sum}(x - 1) \end{aligned}$$

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(plus an infinite set of rules generally omitted)

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  $\Longrightarrow$ we need some way to keep track of "x gets closer to 0"

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- integer TRSs
- $\mathbb{Z}$-TRSs
- (Nagoya) Constrained TRSs

$\mathbb{Z}$-TRSs: Ideas

- Pre-defined symbols: $0, 1 : \text{int},\ +, - : [\text{int} \times \text{int}] \Rightarrow \text{int}$
- User-given symbols with sorts (e.g. sum)
- Constraints using symbols $=, >, \geq, \neg, \wedge$, variables and int-symbols

$\mathbb{Z}$-TRSs: Example

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Reduces for instance $\mathsf{sum}(1+1)$ to $(1+1) + ((1+1)-1) + 0$.

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- technical restrictions on reduction
  (e.g. for a rule $\mathsf{f}(x) \to x + 1$, a term $\mathsf{f}(\mathsf{f}(0))$ can only be reduced innermost)

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(Nagoya) Constrained TRSs: Ideas

- arbitrary underlying set (although usually integers)

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Rewrites e.g. $\mathsf{sum}(\mathsf{s}(\mathsf{s}(0)))$ to $\mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$.

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- "basic" operations not basic
  $\Longrightarrow$ impractical notion of complexity

Observations

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- almost all defined only for integers
- incompatible formalisms; results do not transfer

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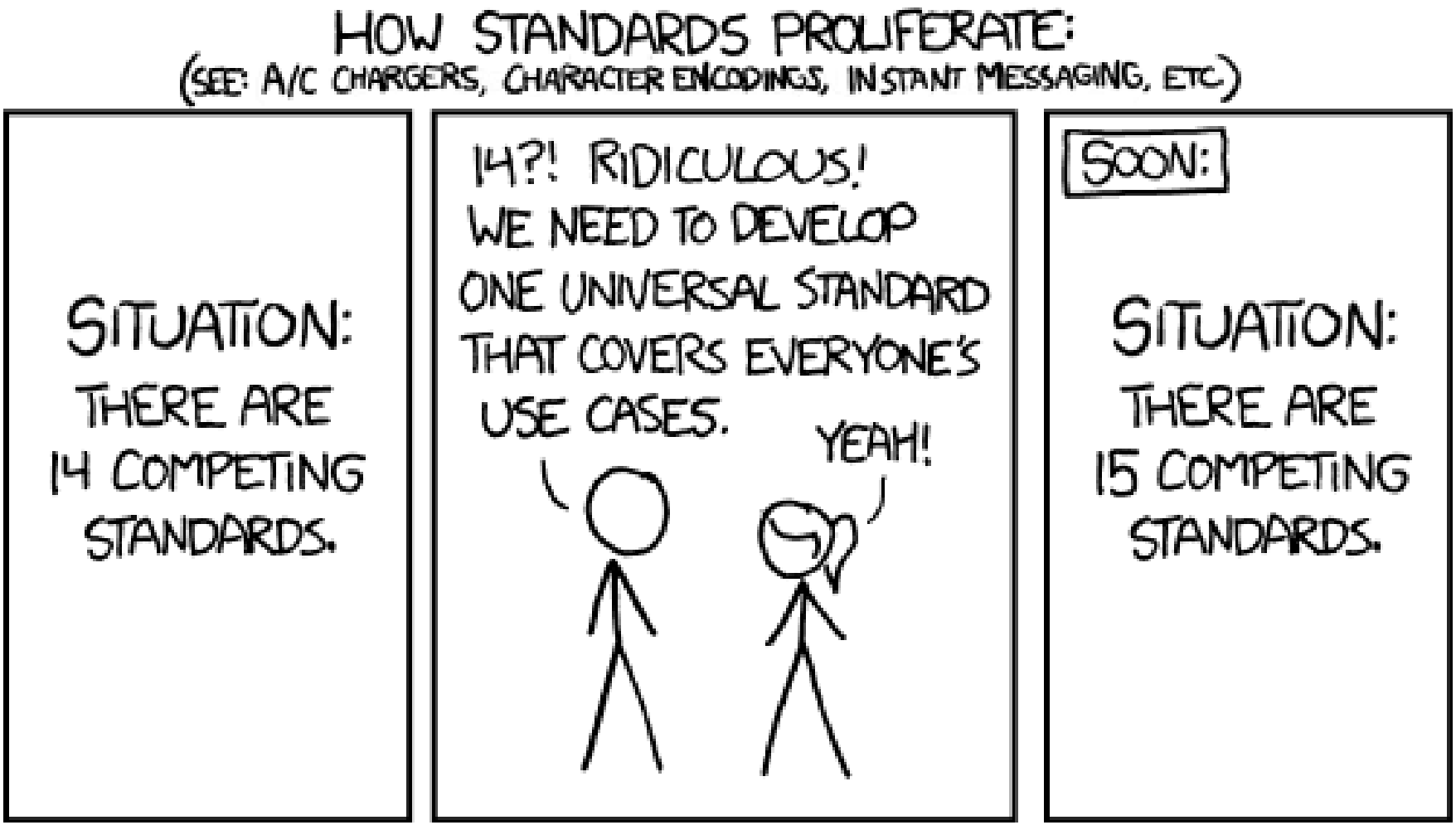
Define constrained rewriting in a natural way,which is general enough for integers and other models (e.g. real numbers, bit vectors), powerful enough to do at least all we can do with the existing formalisms and sufficiently flexible that we can still obtain useful results.



Source: http://www.xkcd.com

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- values represent the elements of the underlying sets
  e.g. $0, 1, 2, \ldots, \text{true}, \text{false}$

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  e.g. $0, 1, \ldots : \text{int},\ +\ : [\text{int} \times \text{int}] \Rightarrow \text{int},\ \text{sum} : [\text{int}] \Rightarrow \text{int},$
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- theory symbols and sorts have a meaning in some fixed model
  e.g. int corresponds to $\mathbb{Z}$, $0, 1, \ldots, +$ are interpreted as usual
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  e.g. $\text{sum}(0, 1)$ and $\text{concat}(\text{"}\textit{hello}\text{"}, \text{"}\textit{world}\text{"})$, but not
  $\text{concat}(\text{"}\textit{hello}\text{"}, 3)$

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  variables in the constraint must be instantiated by values
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  e.g. $3 + 5 \to 8$ and $\mathit{isprime}(17 + 8) \to \mathit{isprime}(25) \to$ false

Reducing sum

$$\begin{array}{lcll} \mathsf{sum}(x) & \to & 0 & [x \leq 0] \\ \mathsf{sum}(x) & \to & x + \mathsf{sum}(x-1) & [x \geq 0] \end{array}$$

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(Naive) Ackerman Function

$$\begin{array}{rcll} \mathsf{ack}(m,n) & \to & \mathsf{ack}(m-1,\mathsf{ack}(m,n-1)) & [m \neq 0 \wedge n \neq 0] \\ \mathsf{ack}(m,0) & \to & \mathsf{ack}(m-1,1) & [m \neq 0] \\ \mathsf{ack}(0,n) & \to & n+1 & \end{array}$$

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With $1, -, +$ interpreted over 16-bit bitvectors.

Bubblesort

```
swap(arr, i, j) {
  int tmp = arr[i];
  arr[i] = arr[j];
  arr[j] = tmp;

}

bubblesort(arr, n) {
  for (int i = 0; i < n; i++)
    for (int j = i+1; j < n; j++)
      if (arr[i] > arr[j])  swap(arr,i,j);

}
```

Bubblesort

```
swap(arr, i, j) {
  int tmp = arr[i];
  arr = arr[i ↦ arr[j]]
  arr = arr[j ↦ tmp];
  return arr;
}

bubblesort(arr, n) {
  for (int i = 0; i < n; i++)
    for (int j = i+1; j < n; j++)
      if (arr[i] > arr[j]) arr = swap(arr,i,j);
  return arr;
}
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Bubblesort

Sorts: IntArray and Int

$$
\begin{array}{rcll}
\mathsf{bubblesort}(arr, n) & \to & \mathsf{loop1}(arr, n, 0) & \\
\mathsf{loop1}(arr, n, i) & \to & \mathsf{loop2}(arr, n, i, 0) & [i < n] \\
\mathsf{loop1}(arr, n, i) & \to & arr & [i \geq n] \\
\mathsf{loop2}(arr, n, i, j) & \to & \mathsf{loop2}(arr, n, i, j + 1) & \\
 & & [j < n \wedge get(arr, i) \leq get(arr, j)] & \\
\mathsf{loop2}(arr, n, i, j) & \to & \mathsf{loop2}(\mathsf{swap}(arr, i, j), n, i, j + 1) & \\
 & & [j < n \wedge get(arr, i) > get(arr, j)] & \\
\mathsf{loop2}(arr, n, i, j) & \to & \mathsf{loop1}(arr, n, i + 1) & [j \geq n] \\
\mathsf{swap}(arr, i, j) & \to & set(set(arr, i, get(arr, j)), j, get(arr, i)) &
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Terms With Constraints

$$\begin{array}{lcll} \mathsf{sum}(x) & \to & 0 & [x \leq 0] \\ \mathsf{sum}(x) & \to & x + \mathsf{sum}(x-1) & [x \geq 0] \end{array}$$

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$$\mathsf{sum}(x-1)\ [\mathsf{true}]$$

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$$\mathsf{sum}(\mathsf{sum}(3) - 1) \; [\mathsf{true}]$$

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- no encodings: we include values of underlying set

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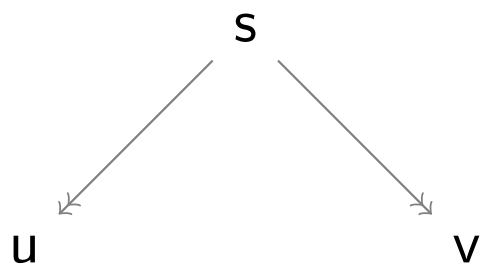
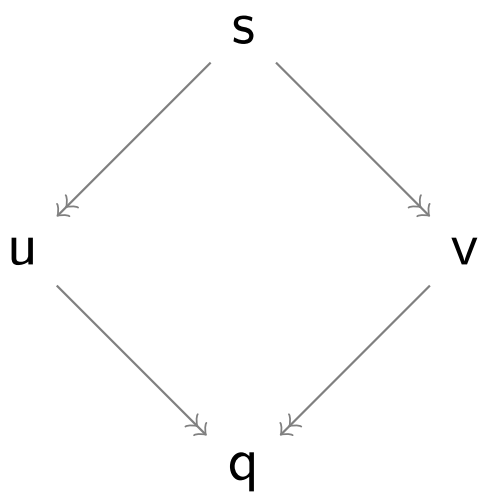
Flexible:

- getting there!

Confluence



Confluence



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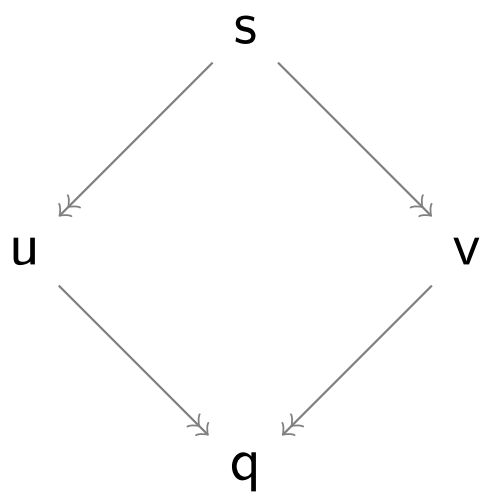

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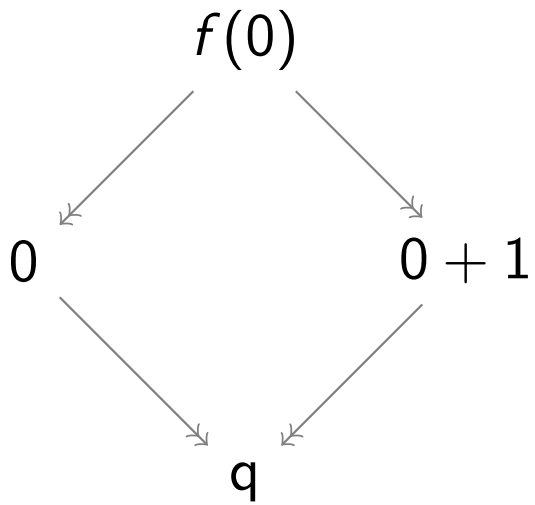

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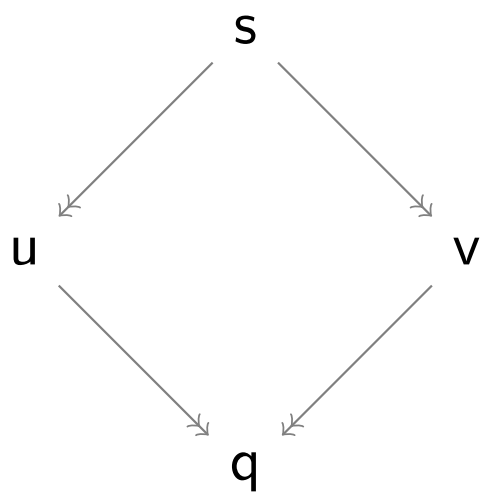

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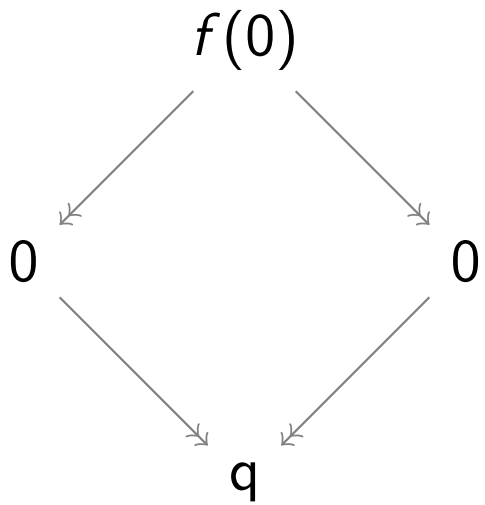

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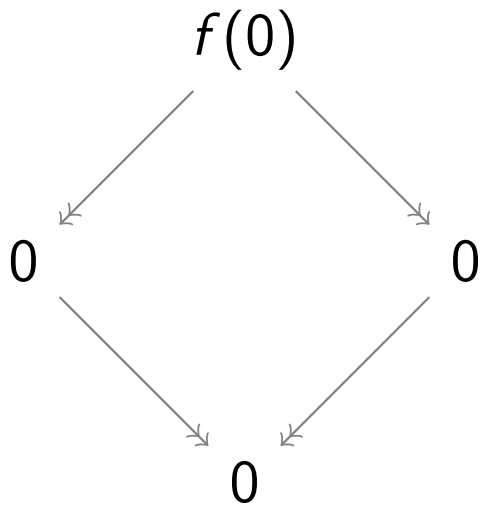

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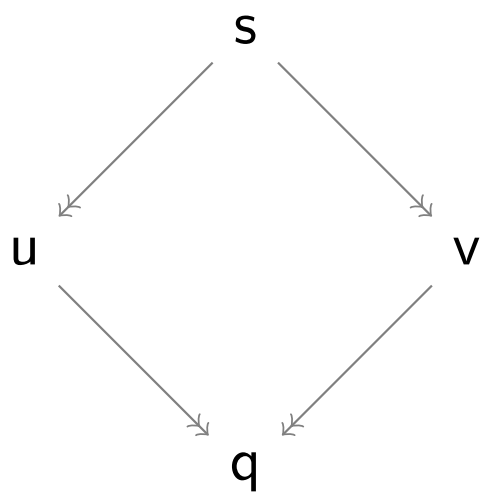

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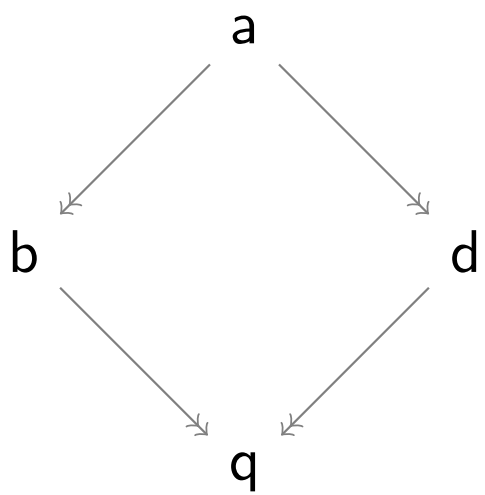

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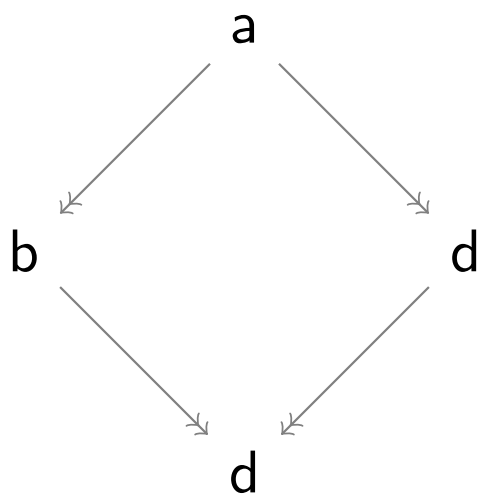

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Orthogonality

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[Klop80]

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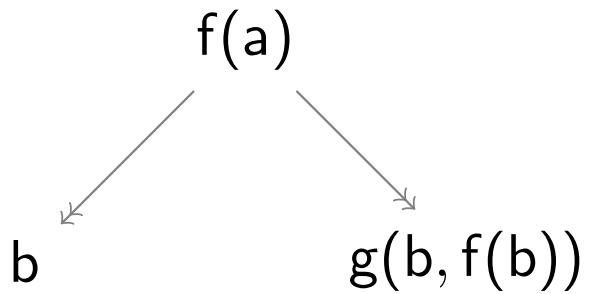

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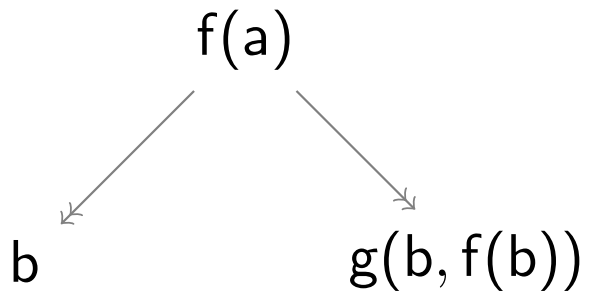

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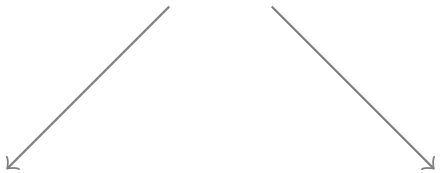

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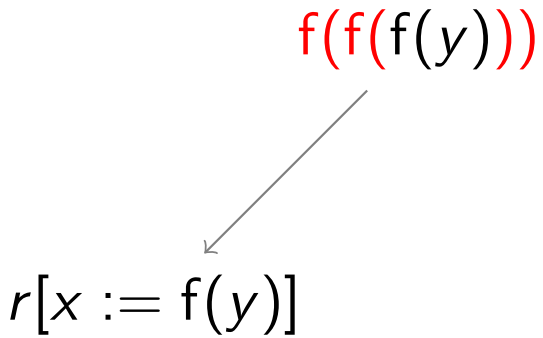

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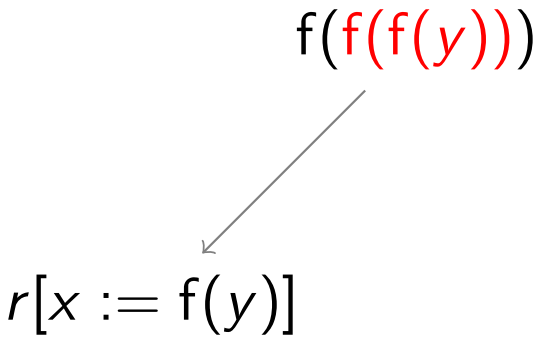

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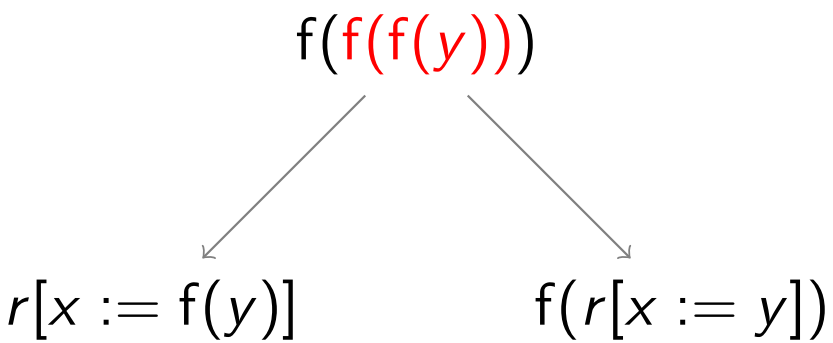

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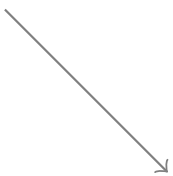

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Solutions:

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- use existing (or future) SMT-solvers!

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int squaresum(int x) {
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