

Finite Local Theory Axiomatizations via Saturation

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Exhaustive Instantiation

$$N \models P(2)$$

$$\forall xy. \phi(x, y)$$

$$\phi(0, 0)$$

$$\phi(0, 1) \quad \phi(1, 0)$$

$$\phi(0, 2) \quad \phi(1, 1) \quad \phi(2, 0)$$

$$\phi(0, 3) \quad \phi(1, 2) \quad \phi(2, 1) \quad \phi(3, 0)$$

\vdots \vdots \vdots \vdots

Locality

$$N \models P(2)$$

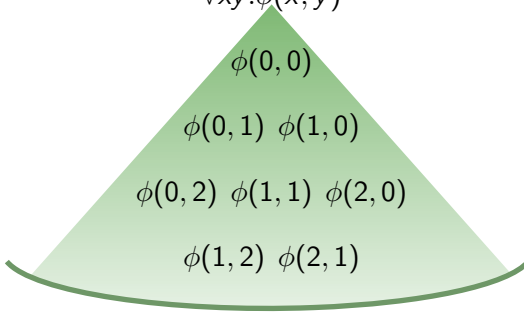
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Recognizing Locality

How to recognize locality?

- ▶ Semantically: Embeddability
- ▶ Syntactically: Saturation / peak saturation

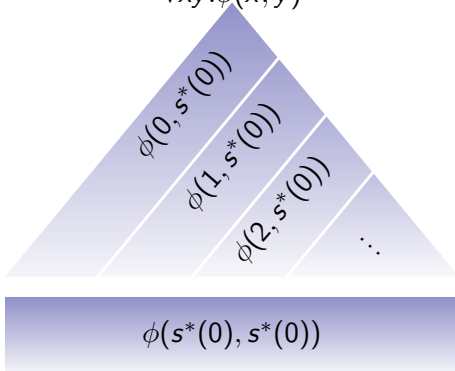
Example Monotonicity:

- ✗ $N_{\leq} \cup \{f(x) \leq f(s(x))\}$
- ✓ $N_{\leq} \cup \{f(x) \leq f(s^n(x)) \mid n \in \mathbb{N}\}$
- ✓ $N_{\leq} \cup \{x \leq y \rightarrow f(x) \leq f(y)\}$

Melting

$$N \models P(2)$$

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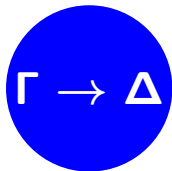


Overview

- ▶ Locality and recognizing it
- ▶ Melting
- ▶ Combining the two

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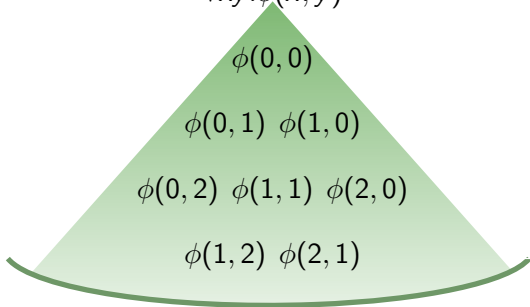
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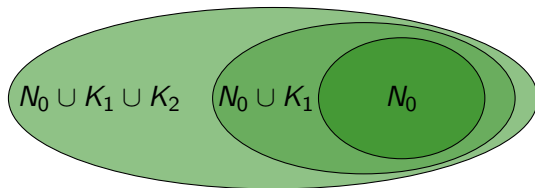
Order Locality

- ▶ Ordering \preceq
- ▶ $\mathbf{N}[\preceq \mathbf{G}] = \{ \text{ground instances of } N \text{ where all terms are } \preceq \text{ some ground } t \text{ in } N \cup G \}$
- ▶ **Order local set:** $N \models G \iff N[\preceq G] \models G$

Locality for Theory Combinations

- ▶ A **theory extension** $N_1 \supseteq N_0$, where $N_1 = N_0 \cup K$, is **order local** if $N_0 \cup K \models G \iff N_0 \cup K[\preceq G] \models G$.
- ▶ Method to reason about different theories step by step.
- ▶ Stacks hierarchically:

$$N_0 \cup K_1 \cup K_2 \models G \iff N_0 \cup K_1 \cup K_2[\preceq G] \models G \iff N_0 \cup K_1 \models G'$$



Recognizing Locality Semantically

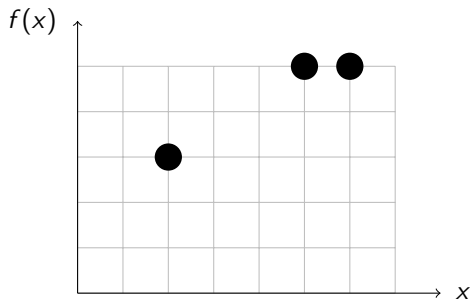
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- ▶ Example: $N_{\leq} \cup \{f(x) \leq f(s(x))\}$

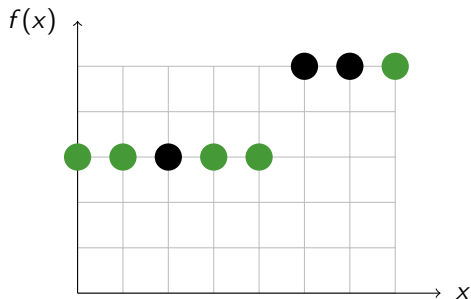
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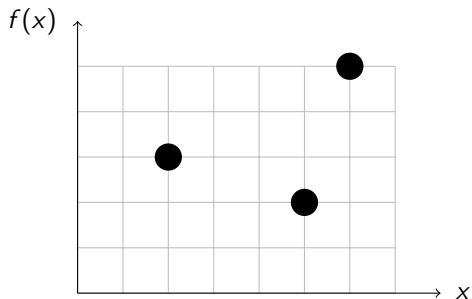
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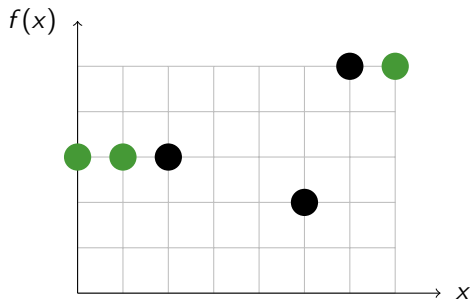
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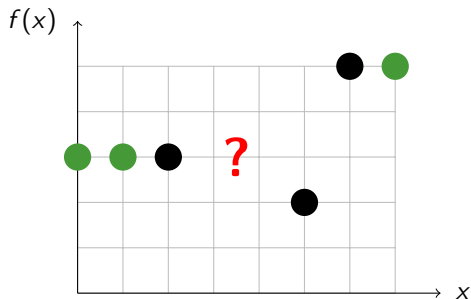
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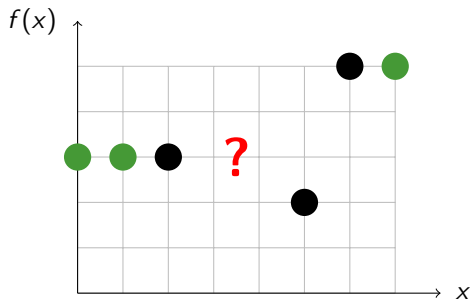
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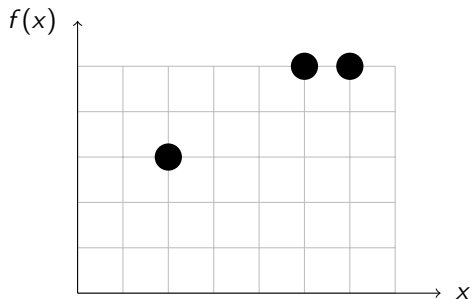
Cannot be extended \implies not local.

Recognizing Locality Semantically

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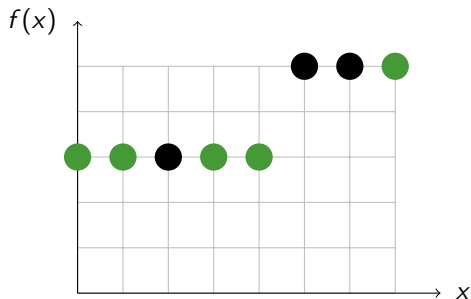
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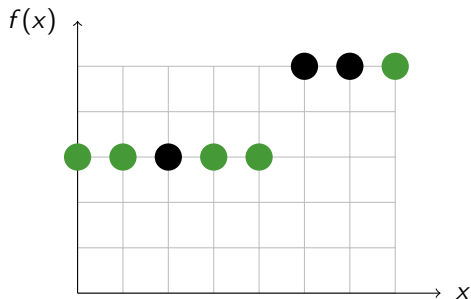
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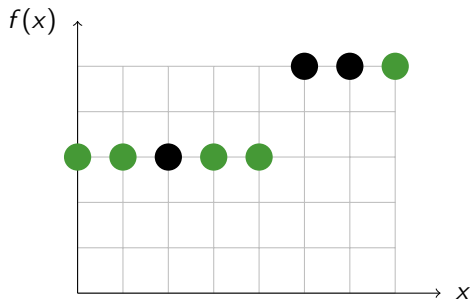
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Every partial model can be extended \implies local.

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- ▶ Example: $N_{\leq} \cup \{x \leq y \rightarrow f(x) \leq f(y)\}$



Every partial model can be extended \implies local.

- ▶ Hard to reason like this automatically.

Recognizing Locality Syntactically

Ordered Resolution:

$$\frac{C, A \quad D, \neg A'}{(C, D)\sigma}$$

where $A\sigma$ is maximal in $(C, A)\sigma$
and $\neg A'\sigma$ is maximal in $(D, \neg A')\sigma$

Example:

$$\frac{f(y) \leq f(s(y)) \quad \frac{f(x) \leq f(s(x)) \quad u \leq v, v \leq w \rightarrow u \leq w}{f(s(x)) \leq w \rightarrow f(x) \leq w}}{f(y) \leq f(s(s(y)))}$$

Saturation: Every resolution inference from N is redundant.

Locality and Saturation (Basin/Ganzinger)

Let

- ▶ $\prec_{\mathcal{T}}$ be a reduction ordering.
- ▶ \prec be compatible and total atom ordering.
- ▶ N be reductive w.r.t. $\prec_{\mathcal{T}}$ and **saturated** w.r.t. \prec ,

Then N is order local.

Peak Saturation (Basin/Ganzinger)

Plateau inference:

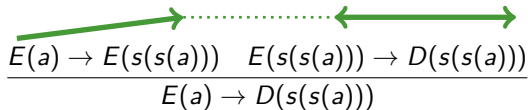
maximal term in succedent of rhs clause is globally maximal

$$\frac{E(a) \rightarrow E(s(s(a))) \quad E(s(s(a))) \rightarrow D(s(s(a)))}{E(a) \rightarrow D(s(s(a)))}$$

Peak Saturation (Basin/Ganzinger)

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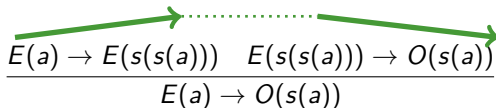


The diagram illustrates the Plateau inference rule. It shows a logical derivation with two premises and one conclusion. The premises are $E(a) \rightarrow E(s(s(a)))$ and $E(s(s(a))) \rightarrow D(s(s(a)))$. The conclusion is $E(a) \rightarrow D(s(s(a)))$. A horizontal dotted line is drawn above the conclusion. A green arrow points from the left end of this line to the right end of the first premise. Another green arrow points from the right end of the dotted line to the left end of the second premise. The entire derivation is enclosed in a fraction-like structure with a horizontal line separating the premises from the conclusion.

$$\frac{E(a) \rightarrow E(s(s(a))) \quad E(s(s(a))) \rightarrow D(s(s(a)))}{E(a) \rightarrow D(s(s(a)))}$$

Peak inference:

premises contain larger terms than the conclusion



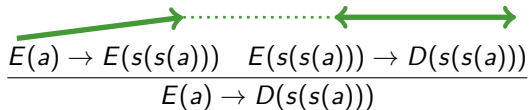
The diagram illustrates the Peak inference rule. It shows a logical derivation with two premises and one conclusion. The premises are $E(a) \rightarrow E(s(s(a)))$ and $E(s(s(a))) \rightarrow O(s(a))$. The conclusion is $E(a) \rightarrow O(s(a))$. A horizontal dotted line is drawn above the conclusion. A green arrow points from the left end of this line to the right end of the first premise. Another green arrow points from the right end of the dotted line to the right end of the second premise. The entire derivation is enclosed in a fraction-like structure with a horizontal line separating the premises from the conclusion.

$$\frac{E(a) \rightarrow E(s(s(a))) \quad E(s(s(a))) \rightarrow O(s(a))}{E(a) \rightarrow O(s(a))}$$

Peak Saturation (Basin/Ganzinger)

Plateau inference:

maximal term in succedent of rhs clause is globally maximal

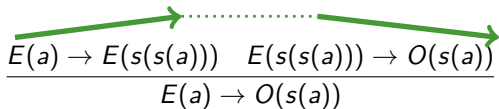


The diagram illustrates the plateau inference rule. It shows a fraction where the numerator consists of two clauses: $E(a) \rightarrow E(s(s(a)))$ on the left and $E(s(s(a))) \rightarrow D(s(s(a)))$ on the right. A horizontal dotted line is drawn above the succedent of the right-hand side clause, $D(s(s(a)))$. A green arrow points from the left towards this dotted line, and another green arrow points from the right towards it, indicating that $D(s(s(a)))$ is the globally maximal term in the succedent of the right-hand side clause. The denominator of the fraction is $E(a) \rightarrow D(s(s(a)))$.

$$\frac{E(a) \rightarrow E(s(s(a))) \quad E(s(s(a))) \rightarrow D(s(s(a)))}{E(a) \rightarrow D(s(s(a)))}$$

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The diagram illustrates the peak inference rule. It shows a fraction where the numerator consists of two clauses: $E(a) \rightarrow E(s(s(a)))$ on the left and $E(s(s(a))) \rightarrow O(s(a))$ on the right. A horizontal dotted line is drawn above the succedent of the right-hand side clause, $O(s(a))$. A green arrow points from the left towards this dotted line, and another green arrow points from the right towards it, indicating that $O(s(a))$ is the globally maximal term in the succedent of the right-hand side clause. The denominator of the fraction is $E(a) \rightarrow O(s(a))$.

$$\frac{E(a) \rightarrow E(s(s(a))) \quad E(s(s(a))) \rightarrow O(s(a))}{E(a) \rightarrow O(s(a))}$$

Peak saturation: Reachable peaks are redundant.

Locality and Peak Saturation (Basin/Ganzinger)

Let

- ▶ $\prec_{\mathcal{T}}$ be a reduction ordering.
- ▶ \prec be compatible and total atom ordering.
- ▶ N be reductive w.r.t. $\prec_{\mathcal{T}}$ and saturated w.r.t. \prec or **peak saturated and Horn**,

Then N is order local.

Locality and Peak Saturation: Example

Monotonicity:

✗ $N_{\leq} \cup \{f(x) \leq f(s(x))\}$

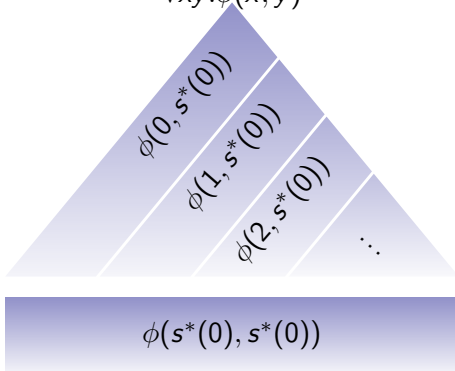
✓ $N_{\leq} \cup \{f(x) \leq f(s^n(x)) \mid n \in \mathbb{N}\}$

✓ $N_{\leq} \cup \{x \leq y \rightarrow f(x) \leq f(y)\}$

Melting

$$N \models P(2)$$

$$\forall xy. \phi(x, y)$$



Constrained Ordered Resolution

$$\frac{C, A \quad D, \neg A'}{(C, D)\sigma}$$

↓

$$\frac{\alpha \parallel C, A \quad \beta \parallel D, \neg A'}{(\alpha, \beta \parallel C, D)\sigma}$$

Constrained Ordered Resolution Example

$$\frac{f(y) \leq f(s(y)) \quad \frac{f(x) \leq f(s(x)) \quad u \leq v, v \leq w \rightarrow u \leq w}{f(s(x)) \leq w \rightarrow f(x) \leq w}}{f(y) \leq f(s(s(y)))}$$

↓

$$\frac{y \approx s(x) \parallel f(x) \leq f(y) \quad \frac{y \approx s(x) \parallel f(x) \leq f(y) \quad u \leq v, v \leq w \rightarrow u \leq w}{y \approx s(x) \parallel f(y) \leq w \rightarrow f(x) \leq w}}{y \approx s(s(x)) \parallel f(x) \leq f(y)}$$

Melting

$$y \approx s(x) \parallel f(x) \leq f(y)$$

$$y \approx s(s(x)) \parallel f(x) \leq f(y)$$

$$y \approx s(s(s(x))) \parallel f(x) \leq f(y)$$

⋮

Melting

$$y \approx_s(x) \parallel f(x) \leq f(y)$$

$$y \approx_{s(s(x))} \parallel f(x) \leq f(y)$$

$$y \approx_{s(s(s(x)))} \parallel f(x) \leq f(y)$$

⋮

$$y \approx_{s^*(s(x))} \parallel f(x) \leq f(y)$$

Melting

$$\begin{aligned}y \approx s(x) \parallel f(x) \leq f(y) \\y \approx s(s(x)) \parallel f(x) \leq f(y) \\y \approx s(s(s(x))) \parallel f(x) \leq f(y) \\ \vdots \\y \approx s^*(s(x)) \parallel f(x) \leq f(y)\end{aligned}$$

$$\frac{\alpha \parallel C \quad \alpha\sigma \parallel C}{\alpha\sigma^* \parallel C}$$

Melting

$$\frac{\alpha \parallel C \quad \alpha\sigma \parallel C}{\alpha\sigma^* \parallel C}$$

Example:

$$\frac{y \approx s(x) \parallel f(x) \leq f(y) \quad y \approx s(s(x)) \parallel f(x) \leq f(y)}{y \approx s^*(s(x)) \parallel f(x) \leq f(y)}$$

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More complex constraints:

$$\frac{p(y) \approx s(x) \parallel f(x) \leq f(y) \quad p(p(y)) \approx s(s(x)) \parallel f(x) \leq f(y)}{p^*(p(y)) \approx s^*(s(x)) \parallel f(x) \leq f(y)}$$

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$$\frac{\alpha \parallel C \quad \alpha\sigma \parallel C}{\alpha\sigma^* \parallel C}$$

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More complex constraints:

$$\frac{p(y) \approx s(x) \parallel f(x) \leq f(y) \quad p(p(y)) \approx s(s(x)) \parallel f(x) \leq f(y)}{(y \approx x)\sigma\sigma^* \parallel f(x) \leq f(y)}$$

where $\sigma = \{ x \mapsto s(x), y \mapsto p(y) \}$

Melting

$$\frac{\alpha \parallel C \quad \alpha\sigma \parallel C}{\alpha\sigma^* \parallel C}$$

Take care of

- ▶ renaming variants of C
- ▶ strategy to propagate the new constraint
- ▶ $*$ in α

$$\frac{x \approx (ss)^*(0) \parallel \text{Even}(x) \quad x \approx pp(ss)^*(0) \parallel \text{Even}(x)}{x \approx (ss|pp)^*(0) \parallel \text{Even}(x)}$$

- ▶ Not always sound!

Melting: Soundness and Completeness

$$\frac{\alpha \parallel C \quad \alpha\sigma \parallel C}{\alpha\sigma^* \parallel C}$$

Unsound:

$$\frac{x \approx 0 \parallel \text{Even}(x) \quad x \approx s(s(0)) \parallel \text{Even}(x)}{x \approx (ss)^*(0) \parallel \text{Even}(x)}$$

Undecidable/Incomplete:

$$(pp)^*(s^*(0)) \approx (ss)^*(0) \parallel \text{false}$$

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- ▶ Premises must be derived from each other
- ▶ Derivation must be repeatable (Horbach/Weidenbach'09)

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Undecidable/Incomplete:

$$(pp)^*(s^*(0)) \approx (ss)^*(0) \parallel \text{false}$$

- ▶ Syntactic criterion: increasing constraints

Melting: Soundness and Completeness

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Decidable/Relatively Complete:

$$(pp)^*(s^*(0)) \approx (ss)^*(0) \parallel \text{false}$$

- ▶ Syntactic criterion: increasing constraints

Example: Monotonicity

- ▶ $N_{\leq} \cup \{f(x) \leq f(s(x))\}$ (not local)
- ▶ **Saturation:** $N_{\leq} \cup \{f(x) \leq f(s^n(x)) \mid n \in \mathbb{N}\}$ (local)
- ▶ Melting is sound and decidable:
- ▶ **With melting:** $N_{\leq} \cup \{y \approx s^*(s(x)) \parallel f(x) \leq f(y)\}$ (loc., finite)
- ▶ **Decide** whether $f(2) \not\leq f(5)$ is possible:

$$\frac{y \approx s^*(s(x)) \parallel f(x) \leq f(y) \quad f(2) \leq f(5) \rightarrow}{5 \approx s^*(s(2)) \parallel \text{false}}$$

- ▶ Constraint satisfiable \implies contradiction found

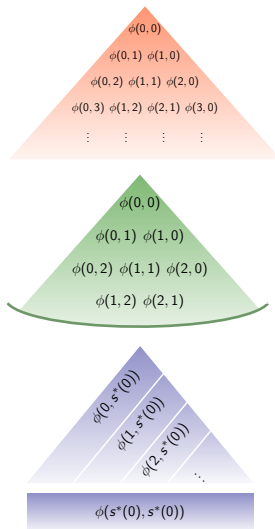
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In this talk:

- ▶ Finite local theory
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- ▶ Locality via (peak) saturation
- ▶ Finiteness via melting

Additionally in the paper:

- ▶ Equations
- ▶ Local theory extensions



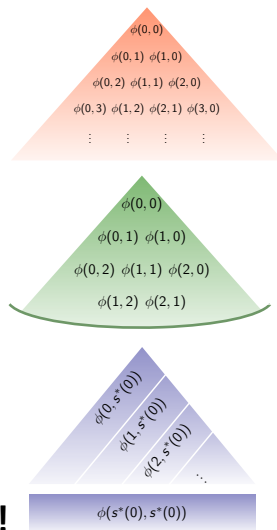
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