Finite Local Theory Axiomatizations via Saturation

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Exhaustive Instantiation

$$N \models P(2)$$

$$\forall xy.\phi(x, y)$$

$$\phi(0,0)$$

$$\phi(0,1) \quad \phi(1,0)$$

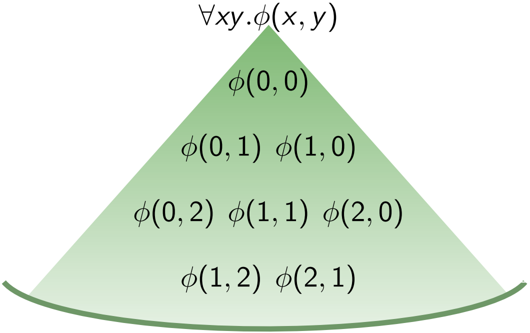
$$\phi(0,2) \quad \phi(1,1) \quad \phi(2,0)$$

$$\phi(0,3) \quad \phi(1,2) \quad \phi(2,1) \quad \phi(3,0)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Locality

$$N \models P(2)$$



Recognizing Locality

How to recognize locality?

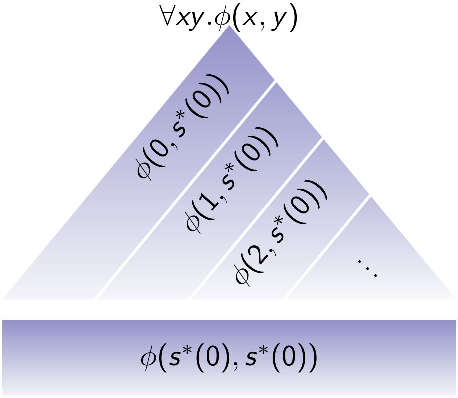
- Semantically: Embeddability
- Syntactically: Saturation / peak saturation

Example Monotonicity:

- × $N_{\leq} \cup \{f(x) \leq f(s(x))\}$
- ✓ $N_{\leq} \cup \{f(x) \leq f(s^n(x)) \mid n \in \mathbb{N}\}$
- ✓ $N_{\leq} \cup \{x \leq y \rightarrow f(x) \leq f(y)\}$

Melting

$$N \models P(2)$$



Overview

- Locality and recognizing it
- Melting
- Combining the two

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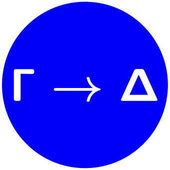

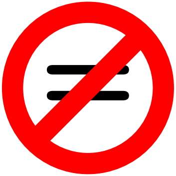

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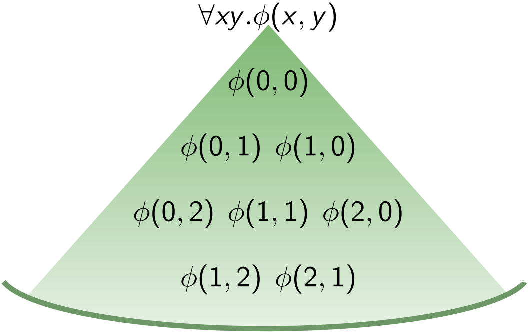


Order Locality

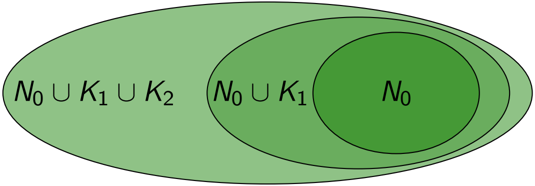
- Ordering $\preceq$
- **N[$\preceq$G]** $= \{$ ground instances of N
  where all terms are $\preceq$ some ground t in $N \cup G$ $\}$
- **Order local set**: $N \models G \iff N[\preceq G] \models G$

Locality for Theory Combinations

- A **theory extension** $N_1 \supseteq N_0$, where $N_1 = N_0 \cup K$, is **order local** if $N_0 \cup K \models G \iff N_0 \cup K[\preceq G] \models G$.

- Method to reason about different theories step by step.
- Stacks hierarchically:

$$N_0 \cup K_1 \cup K_2 \models G \iff N_0 \cup K_1 \cup K_2[\preceq G] \models G \iff N_0 \cup K_1 \models G'$$



Recognizing Locality Semantically

- N is local iff every weak partial model embeds into a total model.

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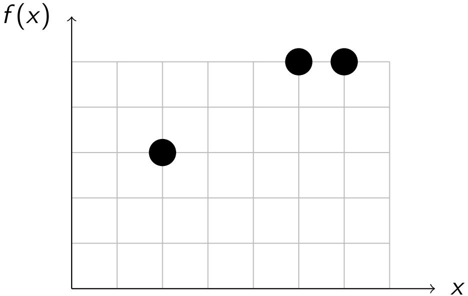

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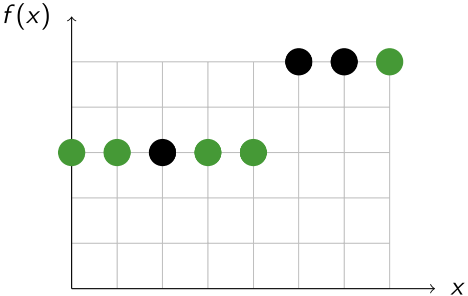

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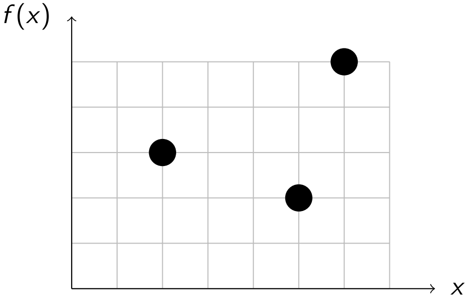

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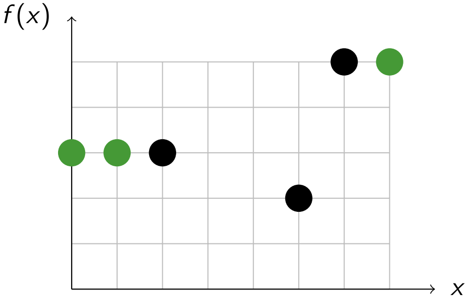

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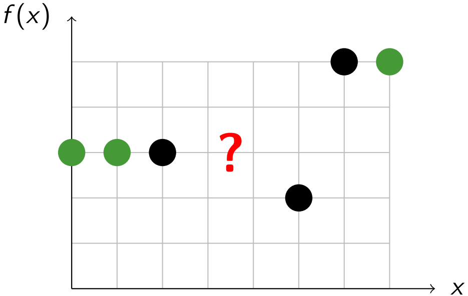

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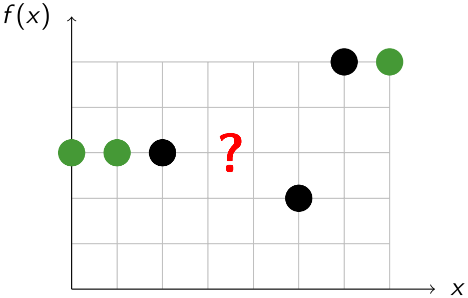

Cannot be extended $\implies$ not local.

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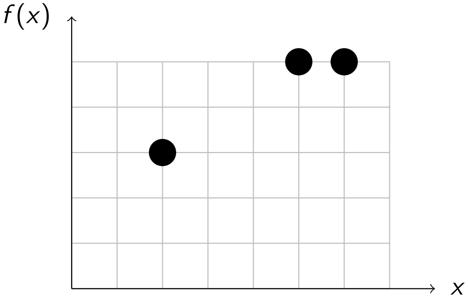

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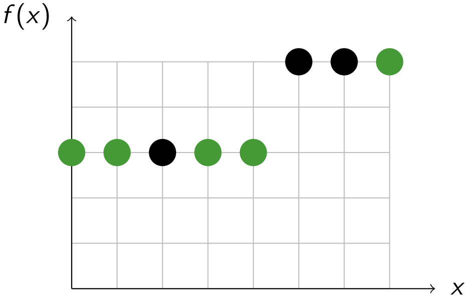

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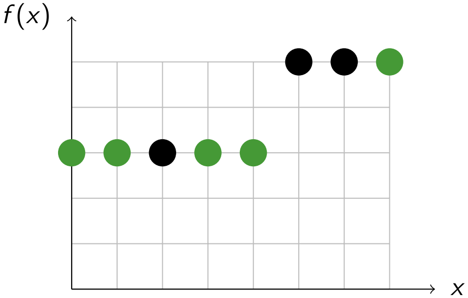

Every partial model can be extended $\implies$ local.

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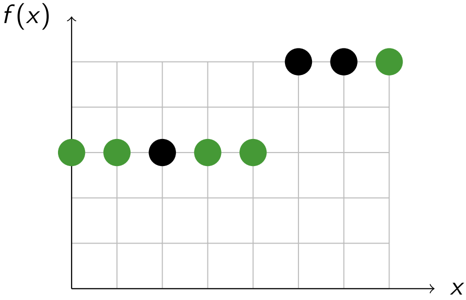

Every partial model can be extended $\Longrightarrow$ local.

- Hard to reason like this automatically.

Recognizing Locality Syntactically

Ordered Resolution:

$$\frac{C, A \qquad D, \neg A'}{(C, D)\sigma}$$

where $A\sigma$ is maximal in $(C, A)\sigma$
and $\neg A'\sigma$ is maximal in $(D, \neg A')\sigma$

Example:

$$\frac{f(y) \leq f(s(y)) \qquad \dfrac{f(x) \leq f(s(x)) \quad u \leq v, v \leq w \rightarrow u \leq w}{f(s(x)) \leq w \rightarrow f(x) \leq w}}{f(y) \leq f(s(s(y)))}$$

Saturation: Every resolution inference from N is redundant.

Locality and Saturation (Basin/Ganzinger)

Let

- $\prec_{\mathcal{T}}$ be a reduction ordering.
- $\prec$ be compatible and total atom ordering.
- N be reductive w.r.t. $\prec_{\mathcal{T}}$ and **saturated** w.r.t. $\prec$,

Then N is order local.

Peak Saturation (Basin/Ganzinger)

Plateau inference:

maximal term in succedent of rhs clause is globally maximal

$$\frac{E(a) \rightarrow E(s(s(a))) \quad E(s(s(a))) \rightarrow D(s(s(a)))}{E(a) \rightarrow D(s(s(a)))}$$

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Peak inference:
premises contain larger terms than the conclusion

$$\frac{E(a) \rightarrow E(s(s(a))) \quad E(s(s(a))) \rightarrow O(s(a))}{E(a) \rightarrow O(s(a))}$$

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Peak saturation: Reachable peaks are redundant.

Locality and Peak Saturation (Basin/Ganzinger)

Let

- $\prec_{\mathcal{T}}$ be a reduction ordering.
- $\prec$ be compatible and total atom ordering.
- N be reductive w.r.t. $\prec_{\mathcal{T}}$ and saturated w.r.t. $\prec$ or **peak saturated and Horn**,

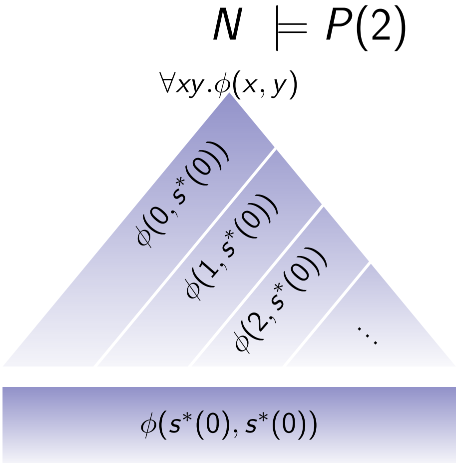
Then N is order local.

Locality and Peak Saturation: Example

Monotonicity:

- × $N_{\leq} \cup \{f(x) \leq f(s(x))\}$
- ✓ $N_{\leq} \cup \{f(x) \leq f(s^n(x)) \mid n \in \mathbb{N}\}$
- ✓ $N_{\leq} \cup \{x \leq y \rightarrow f(x) \leq f(y)\}$

Melting



$N \models P(2)$
$\forall xy.\phi(x, y)$
$\phi(0, s^*(0))$
$\phi(1, s^*(0))$
$\phi(2, s^*(0))$
$\ldots$
$\phi(s^*(0), s^*(0))$

Constrained Ordered Resolution

$$\frac{C, A \qquad D, \neg A'}{(C, D)\sigma}$$

$$\downarrow$$

$$\frac{\alpha \parallel C, A \qquad \beta \parallel D, \neg A'}{(\alpha, \beta \parallel C, D)\sigma}$$

Constrained Ordered Resolution Example

$$\dfrac{f(y) \leq f(s(y)) \qquad \dfrac{f(x) \leq f(s(x)) \quad u \leq v, v \leq w \rightarrow u \leq w}{f(s(x)) \leq w \rightarrow f(x) \leq w}}{f(y) \leq f(s(s(y))}$$

$$\downarrow$$

$$\dfrac{y{\approx}s(x) \parallel f(x) \leq f(y) \qquad \dfrac{y{\approx}s(x) \parallel f(x) \leq f(y) \quad u \leq v, v \leq w \rightarrow u \leq w}{y{\approx}s(x) \parallel f(y) \leq w \rightarrow f(x) \leq w}}{y{\approx}s(s(x)) \parallel f(x) \leq f(y)}$$

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$$y \approx s(x) \parallel f(x) \leq f(y)$$
$$y \approx s(s(x)) \parallel f(x) \leq f(y)$$
$$y \approx s(s(s(x))) \parallel f(x) \leq f(y)$$
$$\vdots$$

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$$\frac{\alpha \parallel C \quad \alpha\sigma \parallel C}{\alpha\sigma^* \parallel C}$$

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Example:

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More complex constraints:

$$\frac{p(y) \approx s(x) \parallel f(x) \leq f(y) \quad p(p(y)) \approx s(s(x)) \parallel f(x) \leq f(y)}{p^*(p(y)) \approx s^*(s(x)) \parallel f(x) \leq f(y)}$$

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More complex constraints:

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where $\sigma = \{\ x \mapsto s(x),\ \ y \mapsto p(y)\ \}$

Melting

$$\frac{\alpha \parallel C \quad \alpha\sigma \parallel C}{\alpha\sigma^* \parallel C}$$

Take care of

- renaming variants of C
- strategy to propagate the new constraint
- $*$ in α

$$\frac{x{\approx}(ss)^*(0) \parallel \mathrm{Even}(x) \quad x{\approx}pp(ss)^*(0) \parallel \mathrm{Even}(x)}{x{\approx}(ss|pp)^*(0) \parallel \mathrm{Even}(x)}$$

- Not always sound!

Melting: Soundness and Completeness

$$\frac{\alpha \parallel C \quad \alpha\sigma \parallel C}{\alpha\sigma^* \parallel C}$$

Unsound:

$$\frac{x{\approx}0 \parallel \mathsf{Even}(x) \quad x{\approx}s(s(0)) \parallel \mathsf{Even}(x)}{x{\approx}(ss)^*(0) \parallel \mathsf{Even}(x)}$$

Undecidable/Incomplete:

$$(pp)^*(s^*(0)){\approx}(ss)^*(0) \parallel \mathsf{false}$$

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- Premises must be derived from each other
- Derivation must be repeatable (Horbach/Weidenbach'09)

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Decidable/Relatively Complete:

$$(pp)^*(s^*(0)){\approx}(ss)^*(0) \parallel \mathrm{false}$$

- Syntactic criterion: increasing constraints

Example: Monotonicity

- $N_{\leq} \cup \{f(x) \leq f(s(x))\}$ (not local)
- **Saturation**: $N_{\leq} \cup \{f(x) \leq f(s^n(x)) \mid n \in \mathbb{N}\}$ (local)
- Melting is sound and decidable:
- **With melting**: $N_{\leq} \cup \{y{\approx}s^*(s(x)) \parallel f(x) \leq f(y)\}$ (loc.,finite)
- **Decide** whether $f(2) \not\leq f(5)$ is possible:

$$\frac{y{\approx}s^*(s(x)) \parallel f(x) \leq f(y) \quad f(2) \leq f(5) \rightarrow}{5{\approx}s^*(s(2)) \parallel \text{false}}$$

- Constraint satisfiable $\implies$ contradiction found

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In this talk:

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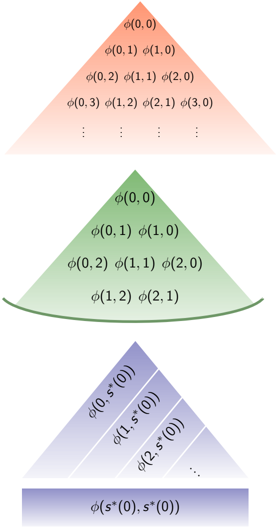
  $\rightsquigarrow$ decision procedure
- Locality via (peak) saturation
- Finiteness via melting

Additionally in the paper:

- Equations
- Local theory extensions



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