Counter Systems: The Quest for Pushing the Decidability Borders

Stéphane Demri

NYU & CNRS – Marie Curie Fellow

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Collaborations

- A survey paper and abstract written with Clark Barrett & Morgan Deters (NYU) in the proceedings.
- Some of the presented results are still subject to on-going research:
  - ACSys group, NYU.
  - Amit Dhar, Arnaud Sangnier (LIAFA).
  - Members of ANR REACHARD (LSV, LABRI).
- Less recent collaborations on the subject with M. Bersani, R. Gascon, V. Goranko, D. D'Souza, R. Lazić, etc.

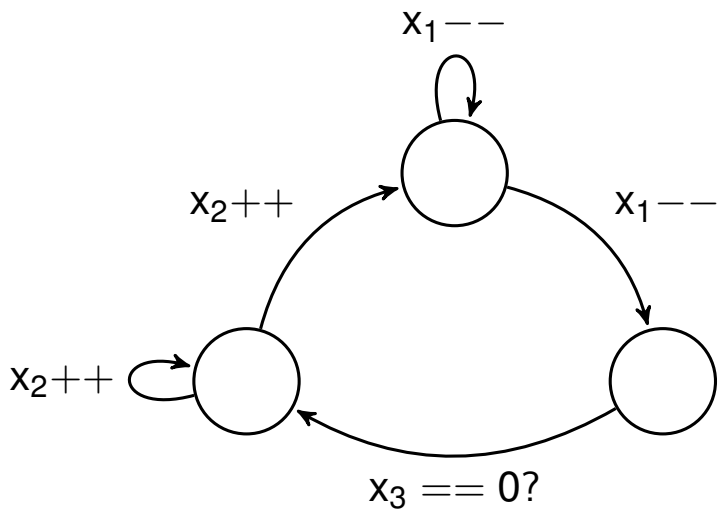
Overview

1. **Presburger Counter Systems**
   - Definition
   - Decision Problems
   - Subclasses of PCS
2. **Loops, Path Schemas and Flatness**
   - Loops & Path Schemas
   - Flatness
3. **Verifying Temporal Properties**
   - Presburger LTL
   - Model-Checking Complexity
4. **Path Schema Subsumption: An Overview**

Presburger Counter Systems

Integer Programs

- Finite-state automaton with counters interpreted by non-negative integers.



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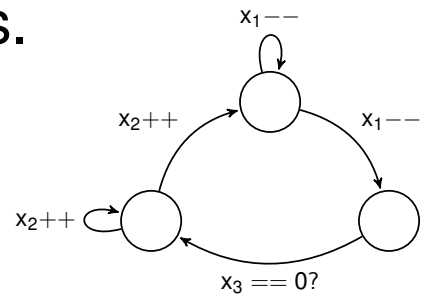

- Many applications:
  - Broadcast protocols, Petri nets, . . .
  - Programs with pointer variables. [Bouajjani et al., CAV'06]
  - Replicated finite-state programs. [Kaiser & Kroening & Wahl, CAV'10]
  - Relationships with data logics. [Bojańczyk et al., TOCL 11]

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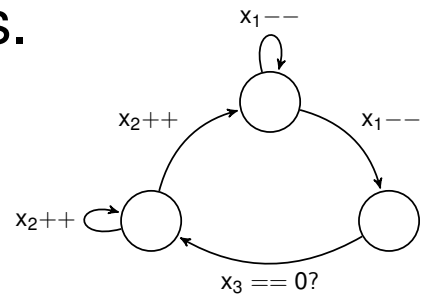

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  - Relationships with data logics. [Bojańczyk et al., TOCL 11]
- Techniques for model-checking infinite-state systems are required for formal verification.
- But, integer programs can simulate Turing machines.
- Checking safety or liveness properties is undecidable.

Taming Verification of Counter Machines

- Design of subclasses with decidable reachability problems

  - Vector addition systems ($\approx$ Petri nets) [Kosaraju, STOC'82]
  - Flat relational counter machines. [Comon & Jurski, CAV'98]
  - Reversal-bounded counter machines. [Ibarra, JACM 78]
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- Decision procedures
  - Translation into Presburger arithmetic. [Fribourg & Olsén, CONCUR'97; Finkel & Leroux, FSTTCS'02]
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  - Well-structured transition systems. [Finkel & Schnoebelen, TCS 01]
- Tools: FAST, LASH, TREX, FLATA, ...

A Fundamental Decidable Theory

- First-order theory of $\langle \mathbb{N}, +, \leq \rangle$ introduced by Mojzesz Presburger (1929).
- Use in guards and in symbolic representations for sets of tuples.
- Many properties: decidability, quantifier elimination, quantifier-free fragment in NP, ...

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- Use in guards and in symbolic representations for sets of tuples.
- Many properties: decidability, quantifier elimination, quantifier-free fragment in NP, ...
- Terms $\mathtt{t} = a_1 \mathsf{x}_1 + \cdots + a_n \mathsf{x}_n + k$ where $a_1, \ldots, a_n \in \mathbb{N}$, k is in $\mathbb{N}$ and the x_i's are variables.
- Presburger formulae: $\phi ::= \mathtt{t} \leq \mathtt{t}' \mid \neg\phi \mid \phi \wedge \phi \mid \exists\, \mathsf{x}\, \phi$

Presburger Arithmetic

- **Linear fragment**: no quantification.

- Valuation $\mathfrak{v} : \mathrm{VAR} \to \mathbb{N}$ + extension to all terms with

$$\mathfrak{v}(a_1 x_1 + \cdots + a_n x_n + k) \stackrel{\text{def}}{=} a_1 \mathfrak{v}(x_1) + \cdots + a_n \mathfrak{v}(x_n) + k$$

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- $\mathfrak{v} \models \mathtt{t} \leq \mathtt{t}'$ iff $\mathfrak{v}(\mathtt{t}) \leq \mathfrak{v}(\mathtt{t}')$; $\mathfrak{v} \models \phi \wedge \phi'$ iff $\mathfrak{v} \models \phi$ and $\mathfrak{v} \models \phi'$,
- $\mathfrak{v} \models \exists \mathrm{x}\ \phi \stackrel{\text{def}}{\Leftrightarrow}$ there is $n \in \mathbb{N}$ such that $\mathfrak{v}[\mathrm{x} \mapsto n] \models \phi$.

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- $\mathfrak{v} \models t \leq t'$ iff $\mathfrak{v}(t) \leq \mathfrak{v}(t')$; $\mathfrak{v} \models \phi \wedge \phi'$ iff $\mathfrak{v} \models \phi$ and $\mathfrak{v} \models \phi'$,
- $\mathfrak{v} \models \exists x\, \phi \stackrel{\text{def}}{\Leftrightarrow}$ there is $n \in \mathbb{N}$ such that $\mathfrak{v}[x \mapsto n] \models \phi$.
- Formula $\phi(x_1, \ldots, x_n)$ with $n \geq 1$ free variables:

$$[\![\phi(x_1, \ldots, x_n)]\!] \stackrel{\text{def}}{=} \{\langle \mathfrak{v}(x_1), \ldots, \mathfrak{v}(x_n)\rangle \in \mathbb{N}^n : \mathfrak{v} \models \phi\}.$$

- ϕ is satisfiable $\stackrel{\text{def}}{\Leftrightarrow}$ there is $\mathfrak{v}$ such that $\mathfrak{v} \models \phi$.

Decision Procedures and Tools

- Quantifier elimination and refinements

  [Cooper, ML 72; Reddy & Loveland, STOC'78]

- Tools dealing with quantifier-free PA, full PA or quantifier elimination: Z3, CVC4, Alt-Ergo, Yices2, Omega test.

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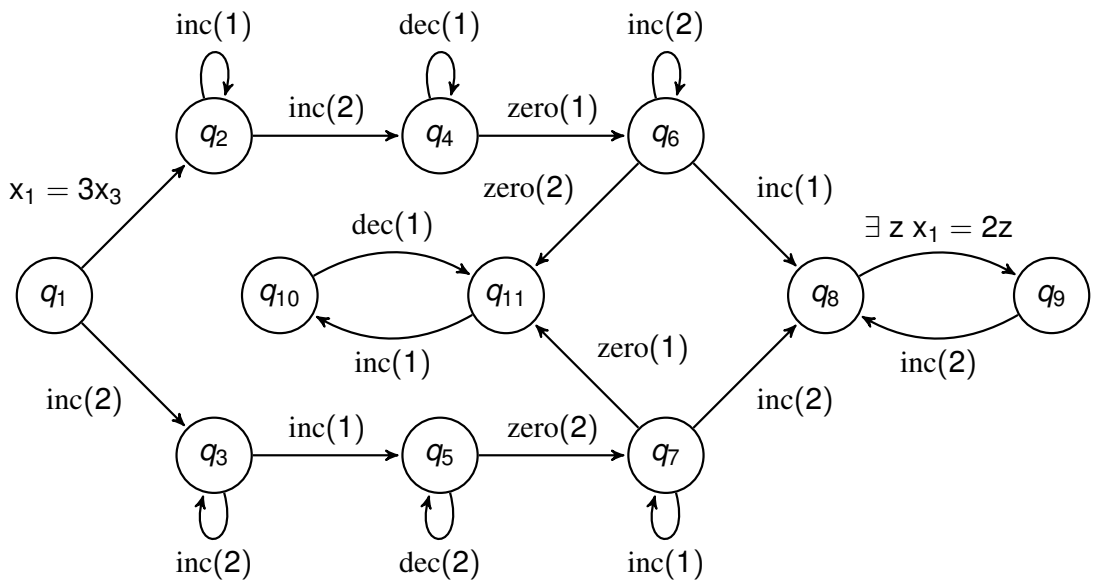
- Automata-based approach.

  [Büchi, ZML 60; Boudet & Comon, CAAP'96]

- Automata-based tools for Presburger arithmetic: LIRA, suite of libraries TAPAS, MONA, and LASH.

Presburger Counter Systems (PCS)

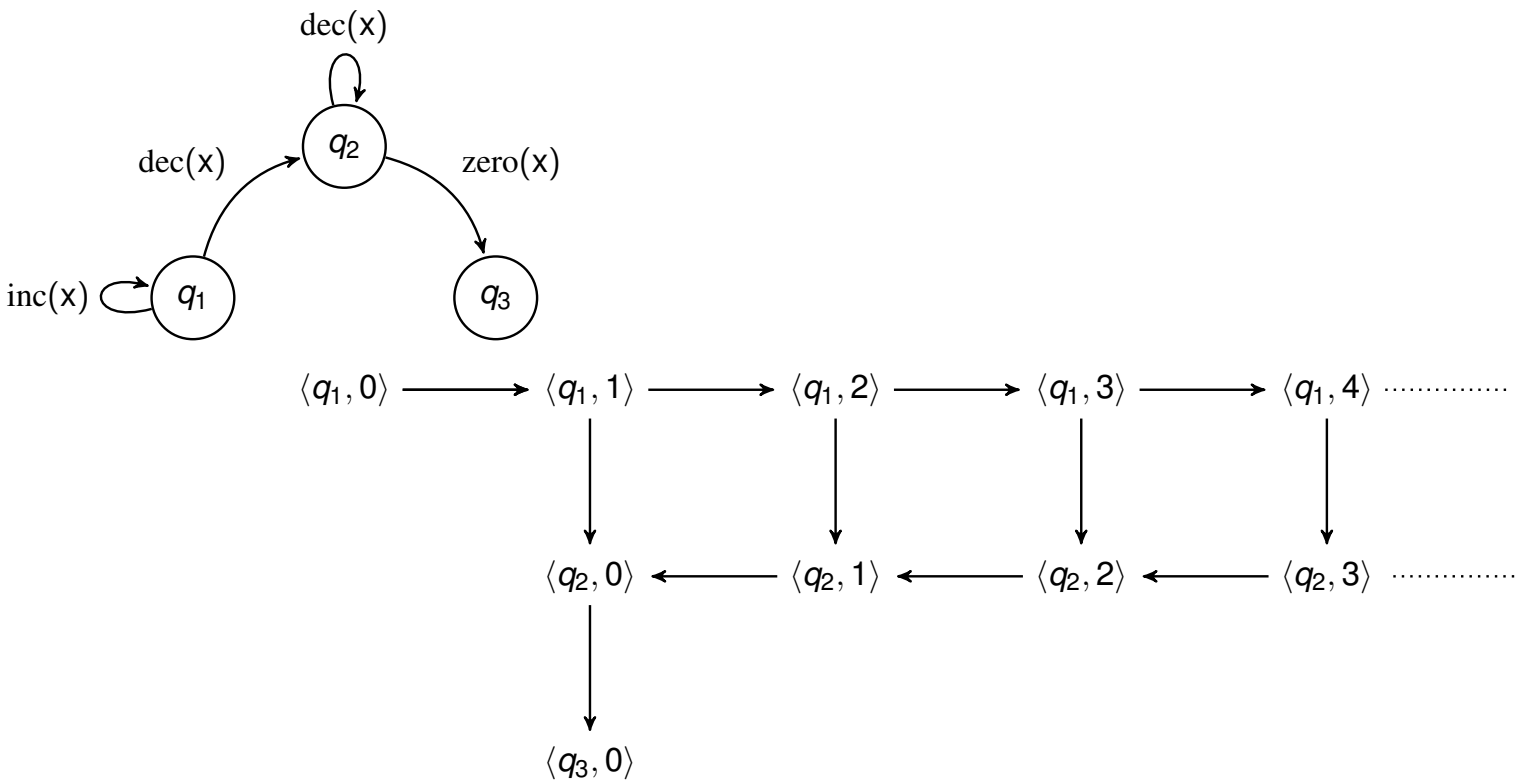
- Presburger Counter System $\mathcal{C} = \langle Q, n, \delta \rangle$ of dimension n:
  - Q is a nonempty finite set of control states.
  - $n \geq 1$ is the dimension.
  - $\delta =$ finite set of transitions of the form $t = \langle q, \phi, q' \rangle$ where $q, q' \in Q$ and ϕ is a Presburger formula with free variables $x_1, \ldots, x_n, x'_1, \ldots, x'_n$.



- Configuration $\langle q, \mathbf{x} \rangle \in \mathfrak{S} = Q \times \mathbb{N}^n$.

Transition System $\mathfrak{T}(\mathrm{C})$

- **Transition system** $\mathfrak{T}(\mathrm{C}) = \langle \mathfrak{S}, \rightarrow \rangle$:
  - $\langle q, \mathbf{x} \rangle \rightarrow \langle q', \mathbf{x'} \rangle \overset{\text{def}}{\Leftrightarrow}$ there is $t = \langle q, \phi, q' \rangle$ such that $\mathfrak{v}[\overline{\mathrm{x}} \leftarrow \mathbf{x}, \overline{\mathrm{x}'} \leftarrow \mathbf{x}'] \models \phi$



- $\overset{*}{\rightarrow}$: reflexive and transitive closure of $\rightarrow$. (sometimes written $\mathtt{Reach}_{\mathrm{C}}$)

Decision Problems

- Reachability problem:

  Input: PCS $\mathbb{C}$, $\langle q_0, \mathbf{x}_0 \rangle$ and $\langle q_f, \mathbf{x}_f \rangle$.

  Question: $\langle q_0, \mathbf{x}_0 \rangle \xrightarrow{*} \langle q_f, \mathbf{x}_f \rangle$?

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- Control state reachability problem:

  Input: PCS $\mathbb{C}$, $\langle q_0, \mathbf{x}_0 \rangle$ and q_f.

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  Input: PCS $\mathbb{C}$, $\langle q_0, \mathbf{x}_0 \rangle$ and q_f.
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- Boundedness problem:
  Input: PCS $\mathtt{C}$ and $\langle q_0, \mathbf{x}_0 \rangle$.
  Question: is $\mathtt{Reach}_{\mathtt{C}}(\langle q_0, \mathbf{x}_0 \rangle)$ finite?

Subclasses of Presburger Counter Systems

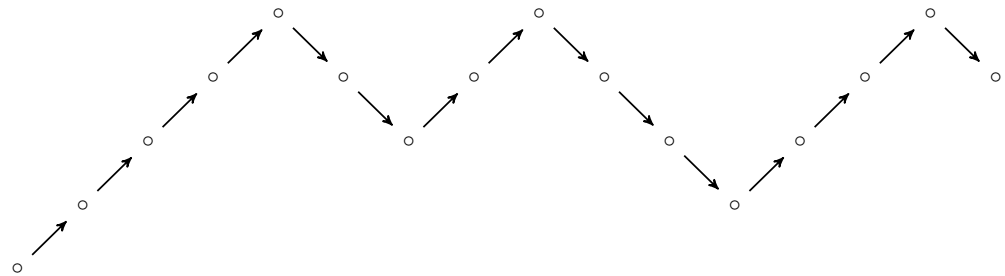
- **Counter systems** (CS): transitions $q \xrightarrow{\phi_g \wedge \phi_u} q' \in \delta$ s.t.
  - ϕ_g is a Boolean combination of atomic formulae of the form $\mathtt{t} \leq \mathtt{t}'$ built over $x_1, \ldots, x_n$ (ex: $2x_1 + x_2 = 3 \vee x_3 \geq 0$),
  - $\phi_u = \bigwedge_{i \in [1,n]} x'_i = x_i + \mathbf{b}(i)$ where $\mathbf{b} \in \mathbb{Z}^n$.
- Minsky machines are counter systems.
- **Vector addition systems with states** (VASS): all the transitions are of the form $q \xrightarrow{\top \wedge \phi_u} q'$.
  ($\approx$ Minsky machines without tests)

Decidability/Complexity Issues for VASS

- The reachability problem is decidable.
  [Mayr, STOC'81; Kosaraju, STOC'82; Leroux, LICS'09]
  - No primitive recursive algorithm is known.
    (use of well quasi-orderings)
  - EXPSPACE-hardness [Lipton, TR 76].
- Boundedness problem for VASS is EXPSPACE-complete.
  [Lipton, TR 76; Rackoff, TCS 78]
- Checking equality between accessibility sets of two configurations is undecidable [Hack, TCS 76].

Reversal-Bounded Counter Systems

- Reversal: Alternation from nonincreasing mode to nondecreasing mode and vice-versa.



- Set $\mathtt{T}$: finite set of terms including $\{x_1, \ldots, x_n\}$.

- Atomic formulae in guards are of the form $\mathtt{t} \leq k$ or $\mathtt{t} \geq k$ with $k \in \mathbb{Z}$ and $\mathtt{t}$ is of the form $\sum_i a_i x_i$ with the a_i's in $\mathbb{Z}$.

- A run is r-$\mathtt{T}$**-reversal-bounded** whenever the number of reversals of each term in $\mathtt{T} \leq r$ times.

Reversal-Boundedness Leads to Semilinearity

- Given a CS $\mathtt{C}$, $\mathtt{T}_\mathtt{C} \stackrel{\text{def}}{=}$ the set of terms $\mathtt{t}$ occurring in $\mathtt{t} \sim k$ with $\sim \in \{\leq, \geq\}$ + counters in $\{\mathtt{x}_1, \ldots, \mathtt{x}_n\}$.

- $\langle \mathtt{C}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ is **reversal-bounded** $\stackrel{\text{def}}{\Leftrightarrow}$ there is $r \geq 0$ such that every run from $\langle q_0, \mathbf{x}_0 \rangle$ is r-$\mathtt{T}_\mathtt{C}$-reversal-bounded.

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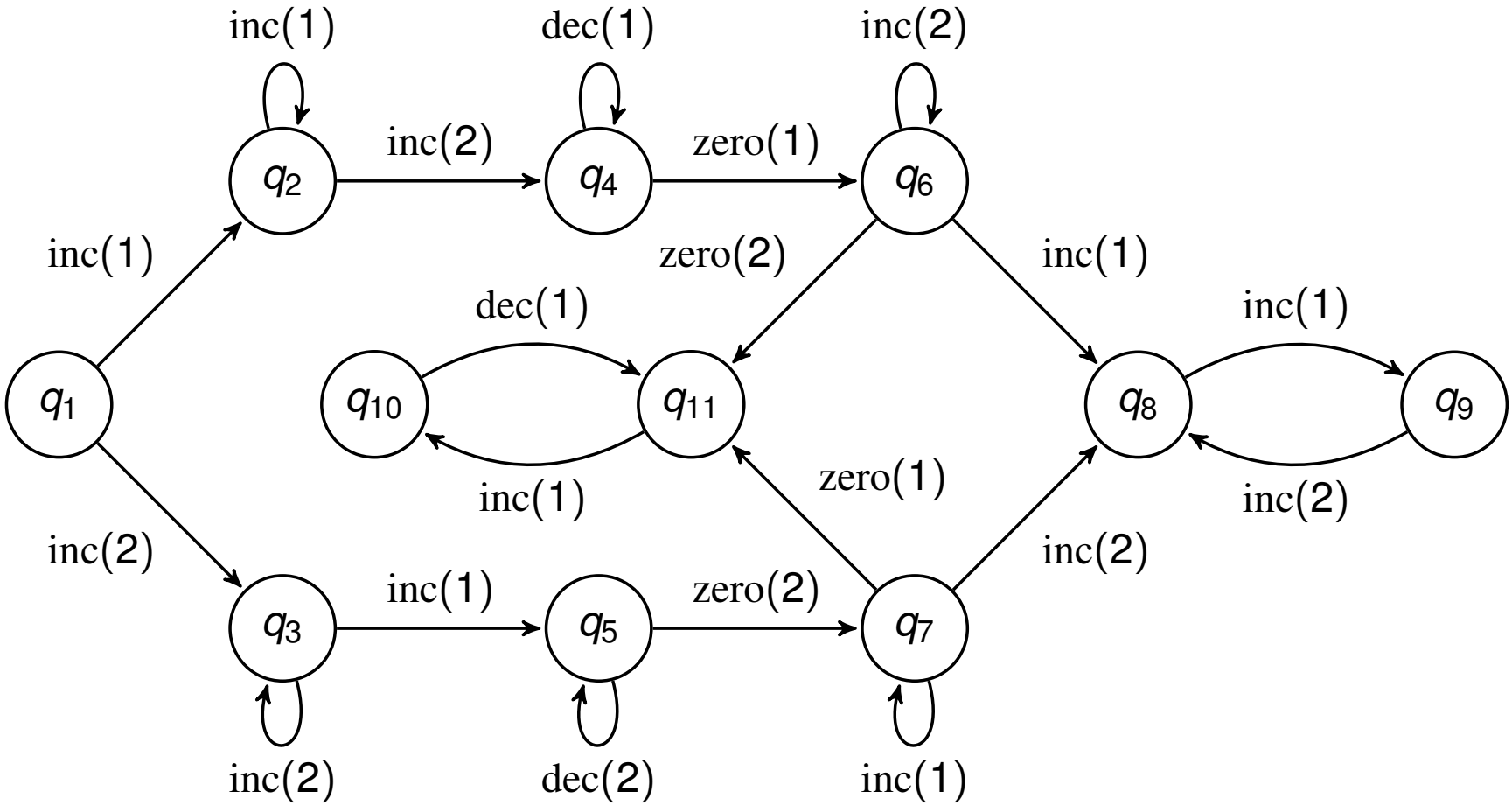
- Given a CS $\mathtt{C}$, $r \geq 0$ and $q, q' \in Q$, one can effectively compute a Presburger formula $\phi_{q,q'}(\overline{\mathtt{x}}, \overline{\mathtt{y}})$ such that for all $\mathfrak{v}$, propositions below are equivalent:
  - $\mathfrak{v} \models \phi_{q,q'}(\overline{\mathtt{x}}, \overline{\mathtt{y}})$,
  - there is an r-$\mathtt{T}_\mathtt{C}$-reversal-bounded run from $\langle q, \langle \mathfrak{v}(\mathtt{x}_1), \ldots, \mathfrak{v}(\mathtt{x}_n) \rangle \rangle$ to $\langle q', \langle \mathfrak{v}(\mathtt{y}_1), \ldots, \mathfrak{v}(\mathtt{y}_n) \rangle \rangle$.

  [Ibarra, JACM 78; Demri & Bersani, FROCOS'11]



$$\phi = (x_1 \geq 2 \wedge x_2 \geq 1 \wedge (x_2 + 1 \geq x_1) \vee (x_2 \geq 2 \wedge x_1 \geq 1 \wedge x_1 + 1 \geq x_2)$$

$$[\![\phi]\!] = \{\mathbf{y} \in \mathbb{N}^2 : \langle q_1, \mathbf{0} \rangle \xrightarrow{*} \langle q_9, \mathbf{y} \rangle\}$$

PCS with Octagonal Constraints

- **Octagonal constraint**: conjunction of atomic guards of the form

$$\pm y \pm z \leq k$$

where y, z are in $x_1, \ldots, x_n, x'_1, \ldots, x'_n$, $k \in \mathbb{Z}$ and $\pm y$ is either y or $-y$.

[Bozga & Girlea & Iosif, TACAS'09]

- **Difference bounds constraint**: conjunction of atomic guards of the form

$$y - z \leq k$$

where y, z are in $x_1, \ldots, x_n, x'_1, \ldots, x'_n$ and $k \in \mathbb{Z}$.

[Comon & Jurski, CAV'98]

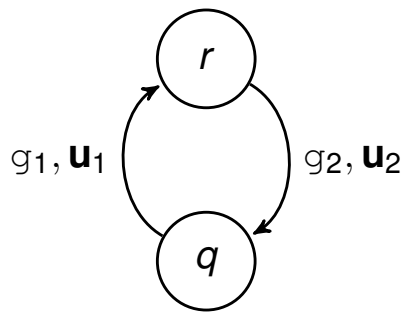
- Guards on transitions in CS are Boolean combinations of linear constraints and therefore are incomparable with the above classes of guards.

Loops, Path Schemas and Flatness

Repeated Loop Effect

- **Path** $\mathtt{p}$: finite sequence of transitions from δ corresponding to a path in its control graph.

- **Loop**: non-empty path $\mathtt{l}$ starting and ending by the same control state.



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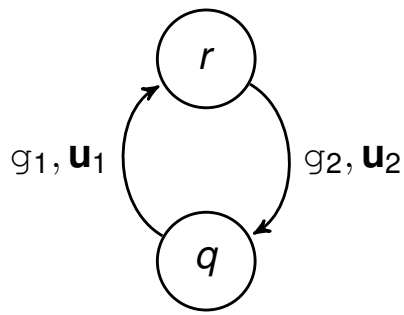

$$\mathtt{l} = t_1 t_2$$

- **Effect** $\mathtt{effect}(\mathtt{l})$:

$$\{\langle \mathbf{x}, \mathbf{x}' \rangle \in \mathbb{N}^n \times \mathbb{N}^n : \langle \mathit{first}(\mathtt{l}), \mathbf{x} \rangle \xrightarrow{\mathtt{l}} \langle \mathit{last}(\mathtt{l}), \mathbf{x'} \rangle\}$$

- **Repeated effect** $\mathtt{effect}^{<\omega}(\mathtt{l})$ (a.k.a **acceleration**):

$$\{\langle \mathbf{x}, \mathbf{x}' \rangle \in \mathbb{N}^n \times \mathbb{N}^n : \langle \mathit{first}(\mathtt{l}), \mathbf{x} \rangle \xrightarrow{\mathtt{l}^i} \langle \mathit{last}(\mathtt{l}), \mathbf{x}' \rangle,\ i \geq 0\}$$

Loop Effects and Presburger Arithmetic

- Reachability problem for loops:

  Input: PCS $\mathtt{C}$, loop $\mathtt{l}$, counter values $\mathbf{x}_0$, $\mathbf{x}_f$.

  question: Is $\langle \mathbf{x}_0, \mathbf{x}_f \rangle \in \mathtt{effect}^{<\omega}(\mathtt{l})$?

- The repeated effect of loops from counter systems is not necessarily definable in Presburger arithmetic.
  (take a loop whose effect is to multiply by 2 a counter)

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- The reachability problem for loops for PCS is undecidable.

- The repeated effect of loops made of octagonal constraints is effectively definable in Presburger arithmetic.

  [Comon & Jurski, CAV'98; Bozga & Girlea &Iosif, TACAS'09]

Counting Iteration

- **Counting iteration** of $R \subseteq \mathbb{N}^{2n}$: $R_{\mathbf{CI}} \subseteq \mathbb{N}^n \times \mathbb{N} \times \mathbb{N}^n$ s.t. $\langle \mathbf{x}, i, \mathbf{y} \rangle \in R_{\mathbf{CI}} \overset{\text{def}}{\Leftrightarrow}$ **y** can be reached from **x** in i steps.
- R has a Presburger counting iteration if its counting iteration is Presburger-definable.
- $\langle \mathbf{x}, \mathbf{y} \rangle \in R^*$ iff there is $i \in \mathbb{N}$ such that $\langle \mathbf{x}, i, \mathbf{y} \rangle \in R_{\mathbf{CI}}$.

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- $\{\langle \alpha, \alpha + 1 \rangle \in \mathbb{N}^2 : \alpha \in \mathbb{N}\}$ has a Presburger counting iteration.
- $R = \{\langle \alpha, 2\alpha \rangle \in \mathbb{N}^2 : \alpha \in \mathbb{N}\}$ has not.
  ($R^* = \{\langle \alpha, 2^\beta \alpha \rangle \in \mathbb{N}^2 : \alpha, \beta \in \mathbb{N}\}$)

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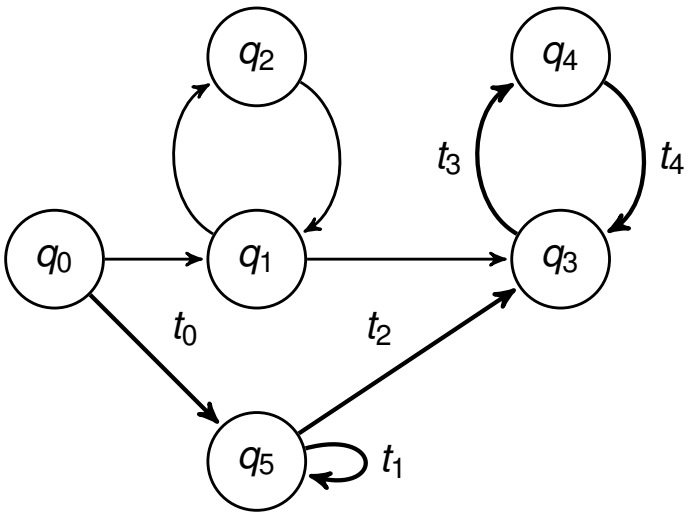
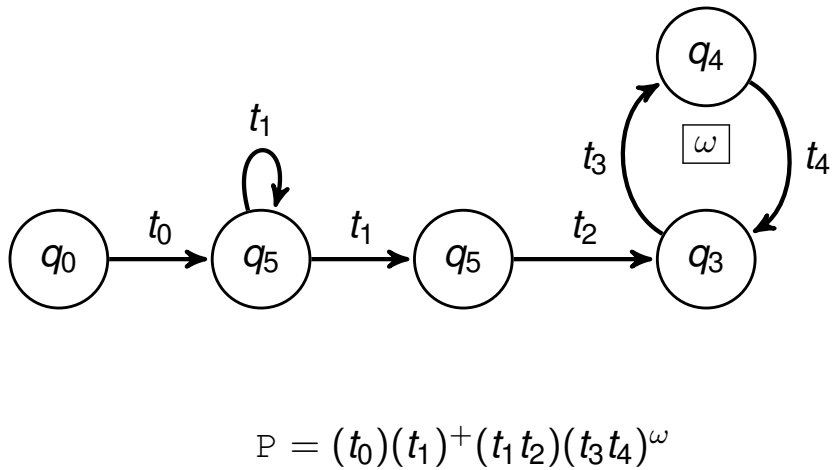
  ($R^* = \{\langle \alpha, 2^\beta \alpha \rangle \in \mathbb{N}^2 : \alpha, \beta \in \mathbb{N}\}$)
- A class of PCS **satisfies the property ($\star$)** when, for every loop `l`, `effect(l)` has the Presburger counting iteration and its Presburger formula is computable.

From Loops to Path Schemas

- Infinitary path schema: alternation of non-loop and loop segments, ending by a loop, and representing a potentially infinite set of infinite runs.





- Infinitary path schema = ω-regular expression of the form $\mathtt{p}_1\mathtt{l}_1^+\mathtt{p}_2\mathtt{l}_2^+\ldots\mathtt{p}_k\mathtt{l}_k^\omega$ over alphabet δ.

- Finitary path schema: $\mathtt{p}_1\mathtt{l}_1^+\mathtt{p}_2\mathtt{l}_2^+\ldots\mathtt{p}_k$ (no last loop).

Runs and Path Schemas

- Run ρ **respects** a path schema when its sequence of transitions belongs to the language of the path schema.
- **Good** path schemas are those that are *minimal* (no loop is multiple of smaller loops, no segment contains a loop,etc.).
- When ($\star$) holds, $\{\langle \mathbf{x}, \mathbf{x}' \rangle : \langle q, \mathbf{x} \rangle \xrightarrow{*} \langle q', \mathbf{x}' \rangle \text{ respects } \mathtt{P}\}$ is effectively Presburger-definable ($\mathtt{P}$ is finitary).

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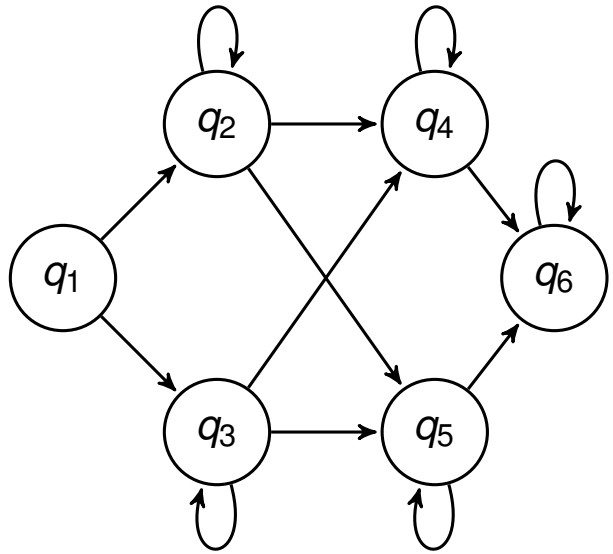
- The class of PCS with octagonal constraints enjoys ($\star$). See also the tool FLATA. [Bozga & Girlea & Iosif, TACAS'09]
- The class CS also enjoys ($\star$).

Flat Presburger Counter Systems

- Every state belongs to at most one simple cycle.

[Fribourg & Olsen, CONCUR'97; Comon & Jurski, CAV'98]



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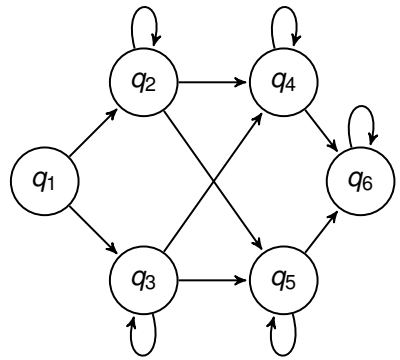

- In a flat counter system, the number of good infinitary path schemas is bounded by $\mathrm{card}(\delta)^{(2\times\mathrm{card}(\delta))}$.
- Let $\mathcal{C}$ be a class of PCS that enjoys $(\star)$. Then, for every flat PCS from $\mathcal{C}$, the relation $\mathtt{Reach}_C$ is Presburger-definable.
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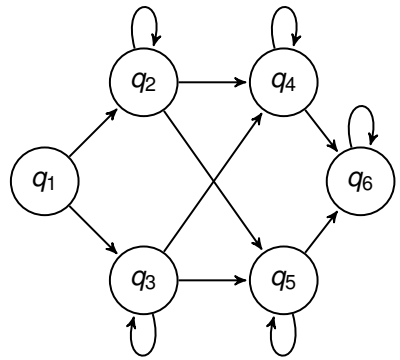

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- Let $\mathcal{C}$ be a class of PCS that enjoys $(\star)$. The reachability problem for $\mathcal{C}$ is decidable.
- The reachability problem for flat CS is NP-complete.

Flattable Systems

- Flat PCS are not always directly available.
- Relaxed version: reachability captured by a flat unfolding.
- $\langle \mathrm{C}, \langle q_0, \mathbf{x}_0 \rangle \rangle$ is **initially flattable** iff there is a finite set of finitary path schemas generating the same configurations.
- Flattable PCS are everywhere. [Leroux & Sutre, ATVA'05]
  - Initialized reversal-bounded CS are initially flattable.
  - Initialized gainy counter automata are initially flattable.
  - Etc.

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  - Initialized reversal-bounded CS are initially flattable.
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  - Etc.
- Flat unfolding of a PCS provides less runs and it can be used as an underapproximation method.
- For VASS, Presburger-definable reachability set iff initially flattable. [Leroux, LICS'13]

We Want More: to Verify Temporal Properties

- Checking how configurations are temporally organized.
- Semilinearity of reachability sets no longer sufficient. How to combine it with other proof techniques?
- Acceleration methods not designed to verify temporal properties.
- How to take advantage of advances in the development of SMT solvers and decision procedures for Presburger arithmetic?

Verifying Temporal Properties

Specifying Existence of Runs in Temporal Logic

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- In temporal logic lingua:

$$\langle q, \mathbf{z}\rangle \models \mathsf{E}\,\exists\, \mathsf{y}_1, \ldots, \mathsf{y}_n\, \mathsf{F}\Big(\bigwedge_{i=1}^{n} \mathsf{x}_i = \mathsf{y}_i \wedge \mathsf{XF}\Big(\bigwedge_{i=1}^{n} \mathsf{x}_i \geq \mathsf{y}_i \wedge \bigvee_{i=1}^{n} \mathsf{x}_i > \mathsf{y}_i\Big)\Big)$$

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- Linear-time temporal logics offer genericity and fragments can be easily designed.

Introduction to Presburger LTL

- First-order quantification over counter values, e.g.

  $\exists\, y\ G(x_1 \leq y). \approx$ "Along the run, counter 1 is bounded."

- $\mathrm{VAR}^p = \{y_1, y_2, \ldots\}$: set of integer variables.

- $\mathrm{VAR} = \{x_1, x_2, \ldots\}$: set of counter variables.

- $Q = \{q_1, q_2, \ldots\}$: set of control state symbols.

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- $Q = \{q_1, q_2, \ldots\}$: set of control state symbols.

- Formulae:

$$\phi ::= \psi \mid q \mid \phi \wedge \phi \mid \neg\phi \mid X\phi \mid \phi U \phi \mid \exists\, y\ \phi$$

  - ψ: Boolean combination of linear constraints over $\mathrm{VAR} \cup \mathrm{VAR}^p$.
  - $q \in Q$.

Satisfaction Relation

- Environment $\mathfrak{E}$: partial map $\mathrm{VAR}^{\mathrm{p}} \to \mathbb{N}$.

$$Q \times \mathbb{N}^n \ni \rho = \langle q_0, \mathbf{x}_0 \rangle, \langle q_1, \mathbf{x}_1 \rangle \cdots \langle q_k, \mathbf{x}_k \rangle \cdots$$

- $\rho, i \models_{\mathfrak{E}} q \overset{\text{def}}{\Leftrightarrow} q = q_i$.
- $\rho, i \models_{\mathfrak{E}} \psi \overset{\text{def}}{\Leftrightarrow} \mathfrak{v}_i \models \psi$ with $\mathfrak{v}_i$ extends $\mathfrak{E}$ s.t. $\mathfrak{v}_i(\mathrm{x}_j) = \mathbf{x_i}(j)$.
- $\rho, i \models_{\mathfrak{E}} \mathrm{X}\phi \overset{\text{def}}{\Leftrightarrow} \rho, i+1 \models_{\mathfrak{E}} \phi$.
- $\rho, i \models_{\mathfrak{E}} \exists\, \mathrm{y}\ \phi$ iff there is $k \in \mathbb{N}$ such that $\rho, i \models_{\mathfrak{E}[\mathrm{y} \mapsto k]} \phi$.

Decision Problems for Presburger LTL

- Semi-closed formula: no variable from $\mathrm{VAR}^{\mathrm{p}}$ is free. $\mathsf{F}(x_1 = y)$ is not semi-closed unlike $\mathsf{G}(x_1 > x_2)$ and $\exists\, y\ \mathsf{G}(x_1 \leq y)$.

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- Satisfiability Problem

  Input: An Presburger LTL semi-closed formula ϕ with free counter variables $\mathsf{x}_1, \ldots, \mathsf{x}_n$.

  Question: Is there a model $\rho \in (\mathbb{Q} \times \mathbb{N}^n)$ s.t. $\rho, 0 \models_\emptyset \phi$?

- Existential Model-Checking Problem

  Input: PCS $\mathcal{C} = \langle Q, n, \delta \rangle$, $\langle q_0, \mathbf{x}_0 \rangle$ and semi-closed formula ϕ with free variables in $\{\mathsf{x}_1, \ldots, \mathsf{x}_n\}$.

  Question: Is there an infinite run ρ starting at $\langle q_0, \mathbf{x}_0 \rangle$ such that $\rho, 0 \models_\emptyset \phi$?

(Infinite runs of PCS are Presburger LTL models)

Temporal Logics with Arithmetical Constraints

- Rich literature:
  - Constraints on the number of event occurrences.
    [Bouajjani et al., LICS'95; Laroussinie et al., TIME'10]
  - Constraints on XML documents.
    [Dal Zilio & Lugiez, RTA'03; Seidl et al., ICALP'04]
  - Temporal semantics of imperative programs.
    [Manna & Pnueli, 1992]

    Program variable x never decreases below its initial value:

    $$\exists y\ (x = y)\ \wedge\ G(x \geq y)$$

  - Graded modal logics.

    See e.g. `http://www.cs.man.ac.uk/~ezolin/ml/`

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- Model-checking restricted to $\mathrm{LTL}(\mathcal{Q})$ is already undecidable.

A Fragment CLTL

- $\psi(\mathsf{X}^{i_1}\mathsf{x}_{j_1},\ldots,\mathsf{X}^{i_k}\mathsf{x}_{j_k})$ as a shortcut for

  $(\exists\, \mathsf{y}_1,\ldots,\mathsf{y}_k\ \mathsf{X}^{i_1}(\mathsf{y}_1=\mathsf{x}_{j_1})\wedge\cdots\wedge\mathsf{X}^{i_k}(\mathsf{y}_k=\mathsf{x}_{j_k})\wedge\psi(\mathsf{y}_1,\ldots,\mathsf{y}_k),$

- $\mathsf{X}^i\mathsf{x}$ understood as the value of x at the *i*th next state.

- First counter remains constant: $\mathsf{G}(\mathsf{x}_1=\mathsf{X}\mathsf{x}_1)$.

- CLTL: fragment of Presburger LTL such that first-order quantification at the level of temporal formulae is restricted to formulae $\psi(\mathsf{X}^{i_1}\mathsf{x}_{j_1},\ldots,\mathsf{X}^{i_k}\mathsf{x}_{j_k})$.

Satisfiability Problems

- Fragment $\mathfrak{F}_0 \ni \phi ::= x_i < x_j \mid x_i = x_j \mid x_i \leq k$.
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- Fragment $\mathfrak{F}_1 \ni \phi ::= x_i \sim x_j + d \mid x_i \sim d$, $d \in \mathbb{Z}$ and $\sim \in \{<, >, \leq, \geq, =\}$.
- Satisfiability problem for $\text{CLTL}^2_1(\mathfrak{F}_1)$ (1 var. + X-length=2) or for $\text{CLTL}^1_2(\mathfrak{F}_1)$ is undecidable.

  See e.g. [Comon & Cortier, CSL'00; Demri & Gascon, TCS 09]

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- Satisfiability problem for $\text{CCTL}^\star(\mathfrak{F}_0)$ (branching-time version of CLTL($\mathfrak{F}_0$)) is decidable. (use of weak MSO with bounding quantifier B)

  [Carapelle & Kartzow & Lohrey, CONCUR'13]

EXPSPACE Upper Bound for VASS

- Control state repeated reachability problem restricted to VASS can be solved in exponential space.
- Model-checking problem restricted to $\mathrm{LTL}(\mathcal{Q})$ and to VASS is EXPSPACE-complete.

  [Habermehl, ICATPN 97]
- Decidability/undecidability results for linear-time temporal logic on Petri nets can be found in [Esparza, CAAP'94]; e.g., $\mathrm{LTL}(\mathcal{Q})$ + $x_i = 0$ is undecidable.

What About Reversal-Bounded Counter Systems?

- Control state repeated reachability problem restricted to reversal-bounded counter systems is decidable.

  See e.g.[Dang & Ibarra & San Pietro, FSTTCS'01]

- Problem RBMC:

  Input: a CS $\mathtt{C}$, $\langle q_0, \mathbf{x}_0 \rangle$, a CLTL formula ϕ, a bound $r \in \mathbb{N}$ (in binary),

  Question: Is there an infinite run ρ from $\langle q_0, \mathbf{x}_0 \rangle$ such that $\rho, 0 \models \phi$ and ρ is r-$\mathtt{T}$-reversal-bounded with $\mathtt{T} = \mathtt{T}_\mathtt{C} \cup \mathtt{T}_\phi$?

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- RBMC is NEXPTIME-complete.

  [Howell & Rosier, JCSS 87]

  [Bersani & Demri, FROCOS'11, Hague & Lin, CAV'11]

  (Proof plan: RBMC $\leq$ repeated reachability $\leq$ reachabillty)

- Global model-checking is also possible for RBMC.

Flat CS and LTL with Arithmetical Constraints

[Demri & Dhar & Sangnier, IJCAR'12]

- Flat CS and arithmetical constraints $\approx$ guards.

$$\phi ::= q \mid g \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \mathsf{X}\phi \mid \phi \mathsf{U} \phi \mid \mathsf{X}^{-1}\phi \mid \phi \mathsf{S} \phi$$

- Model-checking flat CS with this fragment is NP-complete.

Main Ingredients

- An algorithm in NP starts first by guessing a good infinitary path schema

$$\mathtt{p}_1\mathtt{l}_2^+\mathtt{p}_3\mathtt{l}_4^+\cdots\mathtt{l}_{k-1}^+\mathtt{p}_k\mathtt{l}_k^\omega$$

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  3. Stuttering Theorem for Past LTL

How to Internalize the Choices for Path Schemas

- How to deal with the nondeterminism related to the choice of the infinitary path schema using Boolean formulae?
- How to deal with the nondeterminism related to the unfolding of the path schemas for eliminating disjunctions?

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- How to deal with the nondeterminism related to the number of times loops are visited?
- May be related to how to find good accelerations?

  See e.g. [Finkel & Leroux, FSTTCS'02; Gonnord, PhD 06]

Other Logics on Flat CS

- Model-checking flat counter systems with FO or linear μ-calculus is PSPACE-complete.
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- *Open question*: decidability status of model-checking flat CS beyond CTL*.

Path Schema Subsumption: An Overview

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- How to generate path schemas in a structured and controlled fashion?
- How to find a finite set of path schemas that fully captures the behavior of a PCS, if possible?
- Strategy in which we have a clear way of detecting whether we have enumerated sufficiently many path schemas.

Consistency and Subsumption

- Finitary path schemas $\mathtt{P}_1, \ldots, \mathtt{P}_\alpha, \mathtt{P}$. All the path schemas start and end by the same control states.
- **Consistency** of $\mathtt{P}$ wrt the initial condition $\phi_{\text{init}}(y_1, \ldots, y_n)$:

$$\exists\, x_1, \ldots, x_n\ \exists\, x'_1, \ldots, x'_n\ \phi_{\text{init}}(x_1, \ldots, x_n) \wedge$$

$$\varphi_{\mathtt{P}}(x_1, \ldots, x_n, x'_1, \ldots, x'_n)$$

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- Existence of formula $\varphi_{\mathtt{P}}$ guaranteed by property $(\star)$.
- For the class of CS, the consistency problem is NP-complete.
- $\{\mathtt{P}_1, \ldots, \mathtt{P}_\alpha\}$ **subsumes** $\mathtt{P}$ **wrt** $\phi_{\text{init}}(y_1, \ldots, y_n)$:

$$\forall\, x_1, \ldots, x_n\ \forall\, x'_1, \ldots, x'_n\ (\phi_{\text{init}}(x_1, \ldots, x_n) \wedge \varphi_{\mathtt{P}}(x_1, \ldots, x_n, x'_1, \ldots, x'_n))$$

$$\Rightarrow \bigvee_{i \in [1,\alpha]} \exists\, z_1, \ldots, z_n\ \phi_{\text{init}}(z_1, \ldots, z_n) \wedge \varphi_{\mathtt{P}_i}(z_1, \ldots, z_n, x'_1, \ldots, x'_n)$$

General Subsumption (one step beyond reachability)

- $[\![\mathtt{P}]\!]_{\phi_{\text{init}}} \stackrel{\text{def}}{=} \{\mathbf{x}_f : \langle q_0, \mathbf{x}_0 \rangle \stackrel{*}{\rightarrow} \langle q_f, \mathbf{x}_f \rangle \text{ respects } \mathtt{P} \text{ and } \mathbf{x}_0 \models \phi_{\text{init}}\}$.

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- $[\mathtt{P}]_{\phi_{\text{init}}} \stackrel{\text{def}}{=} \{\mathbf{x}_f : \langle q_0, \mathbf{x}_0\rangle \stackrel{*}{\rightarrow} \langle q_f, \mathbf{x}_f\rangle \text{ respects } \mathtt{P} \text{ and } \mathbf{x}_0 \models \phi_{\text{init}}\}$.
- **Pattern** ϕ_{pat}: formula in Presburger LTL without FO quantification and with free occurrences of $\mathsf{y}_1, \ldots, \mathsf{y}_\alpha$.

$$[\mathtt{P}]_{\phi_{\text{pat}},\phi_{\text{init}}} \stackrel{\text{def}}{=} \{\mathfrak{E} : \rho = \langle q_0, \mathbf{x}_0\rangle \stackrel{*}{\rightarrow} \langle q_f, \mathbf{x}_f\rangle \text{ respects } \mathtt{P},$$
$$\mathbf{x}_0 \models \phi_{\text{init}} \;\&\; \rho, 0 \models_{\mathfrak{E}} \phi_{\text{pat}}\}$$

- $[\mathtt{P}]_{\phi_{\text{init}}}$ above corresponds to $[\mathtt{P}]_{\phi_{\text{pat}},\phi_{\text{init}}}$ with

$$\phi_{\text{pat}} \stackrel{\text{def}}{=} \mathsf{F}(\mathsf{x}_1 = \mathsf{y}_1 \wedge \cdots \wedge \mathsf{x}_n = \mathsf{y}_n \wedge \neg \mathsf{X}\top)$$

General Subsumption (one step beyond reachability)

- **Pattern** ϕ_{pat}: formula in Presburger LTL without FO quantification and with free occurrences of $\mathsf{y}_1, \ldots, \mathsf{y}_\alpha$.

$$[\mathtt{P}]_{\phi_{\text{pat}},\phi_{\text{init}}} \stackrel{\text{def}}{=} \{\mathfrak{E} : \rho = \langle q_0, \mathbf{x}_0\rangle \xrightarrow{*} \langle q_f, \mathbf{x}_f\rangle \text{ respects } \mathtt{P},$$
$$\mathbf{x}_0 \models \phi_{\text{init}} \;\&\; \rho, 0 \models_{\mathfrak{E}} \phi_{\text{pat}}\}$$

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- $\{\mathtt{P}_1, \ldots, \mathtt{P}_\alpha\}$ **subsumes** $\mathtt{P}$ **wrt** $\phi_{\text{init}}(\mathsf{y}_1, \ldots, \mathsf{y}_n)$ **and the pattern** ϕ_{pat} $\stackrel{\text{def}}{\Leftrightarrow}$ $[\mathtt{P}]_{\phi_{\text{pat}},\phi_{\text{init}}} \subseteq [\mathtt{P}_1]_{\phi_{\text{pat}},\phi_{\text{init}}} \cup \cdots \cup [\mathtt{P}_\alpha]_{\phi_{\text{pat}},\phi_{\text{init}}}$.
- For any class of PCS satisfying ($\star$), there is a reduction from the generalized path schema subsumption problem to the validity problem for (PA).

How to Deal with Quantifiers

- Subsumption tests contain quantifiers.
- Most well-known Satisfiability Modulo Theories (SMT) solvers deal with quantifier-free formulae.
- Dealing with quantifiers is usually a difficult task for SMT solvers.
- Examples of techniques to deal with quantifiers:
  - Quantifier elimination. See e.g. [Reddy & Loveland, STOC'78]
  - Heuristic instantiation. See e.g. [Reynolds et al, CADE'13]
  - Lazy approach to quantifier elimination. [Monniaux, CAV'10]
- Challenge: How to use SMT solvers to enumerate path schemas and to perform subsumption?

A Few Words about the Enumeration Algorithm

- Paper contains a sketch of the algorithm for enumerating path schemas.
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A Few Words about the Enumeration Algorithm

- Paper contains a sketch of the algorithm for enumerating path schemas.
- Subsumption tests parameterized by patterns and cycles are generated with cycle schemas.
- Generation of path schemas without arithmetical constraints is complete, stratified and takes advantage of the generation of cycle schemas.
- With subsumption on counter values, a complete version of the algorithm can be obtained if cycles are generated independently of cycle schemas.

Concluding Remarks

- Verification of temporal properties on PCS in its infancy.
- Need for methods to deal with (full) Presburger arithmetic or for proof systems dealing with model-checking.
- How to take advantage of recent developments on SAT/SMT solvers to deal with nondeterminism, quantified formulae etc.?

Concluding Remarks

- Verification of temporal properties on PCS in its infancy.
- Need for methods to deal with (full) Presburger arithmetic or for proof systems dealing with model-checking.
- How to take advantage of recent developments on SAT/SMT solvers to deal with nondeterminism, quantified formulae etc.?
- Other related trends include SMT solvers for model-checking infinite-state systems, branching VASS, complexity for VASS, relationships between CS and data logics, etc.

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- Dates
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  - Notification **March 31st, 2014**
  - Conference **July 19th to July 22nd 2014**
  - Affiliated workshops **July 17, 18, 23, 24**