Decidability and complexity via mosaics of the temporal logic of the lexicographic products of unbounded dense linear orders

Philippe Balbiani* and Szabolcs Mikulás**

*Institut de recherche en informatique de Toulouse
CNRS — University of Toulouse
**Department of Computer Science and Information Systems
Birbeck — University of London

Overview

1. **Asynchronous product of normal modal logics**
2. **Lexicographic product of normal modal logics**
3. **Limited hyperreals**
4. **Lexicographic product of linear temporal logics**
5. **Axiomatization/completeness**
6. **Decidability/complexity**
7. **Conclusion and open problems**

Asynchronous product of normal modal logics

Asynchronous product of relational structures

- $\mathcal{F}_1 = (W_1, R_1)$, $\mathcal{F}_2 = (W_2, R_2)$
- $\mathcal{F}_1 \times \mathcal{F}_2 = (W, S_1, S_2)$ where
  - $W = W_1 \times W_2$
  - $(x_1, x_2)S_1(y_1, y_2)$ iff $x_1 R_1 y_1$ and $x_2 = y_2$
  - $(x_1, x_2)S_2(y_1, y_2)$ iff $x_1 = y_1$ and $x_2 R_2 y_2$

Proposition: Let $\mathcal{F} = (W, S_1, S_2)$ be countable and such that

- $S_1 \circ S_2 = S_2 \circ S_1$
- $S_2^{-1} \circ S_1 \subseteq S_1 \circ S_2^{-1}$

There exists $\mathcal{F}_1 = (W_1, R_1)$, $\mathcal{F}_2 = (W_2, R_2)$ such that $\mathcal{F}$ is a bounded morphic image of $\mathcal{F}_1 \times \mathcal{F}_2$

Asynchronous product of normal modal logics

Asynchronous product of relational structures

- $\mathcal{F}_1 = (W_1, R_1)$, $\mathcal{F}_2 = (W_2, R_2)$
- $\mathcal{F}_1 \times \mathcal{F}_2 = (W, S_1, S_2)$ where
  - $W = W_1 \times W_2$
  - $(x_1, x_2) S_1 (y_1, y_2)$ iff $x_1 R_1 y_1$ and $x_2 = y_2$
  - $(x_1, x_2) S_2 (y_1, y_2)$ iff $x_1 = y_1$ and $x_2 R_2 y_2$

Combining normal modal logics: asynchronous product

- $L_1 \times L_2 = Log\{\mathcal{F}_1 \times \mathcal{F}_2 : \mathcal{F}_1 \models L_1$ and $\mathcal{F}_2 \models L_2\}$

Asynchronous product of normal modal logics

Product matching normal modal logics

- L_1 and L_2 are $\times$-product matching iff
  $L_1 \times L_2 = L_1 \otimes L_2 \otimes \Diamond_1\Diamond_2 p \leftrightarrow \Diamond_2\Diamond_1 p \otimes \Diamond_1\Box_2 p \rightarrow \Box_2\Diamond_1 p$

Proposition: Let L_1 and L_2 be normal modal logics from the following list

- K, D, T, $K4$, $D4$, $S4$, $K45$, $KD45$, $S5$

L_1 and L_2 are $\times$-product matching

Lexicographic product of normal modal logics

Lexicographic product of relational structures

- $\mathcal{F}_1 = (W_1, R_1)$, $\mathcal{F}_2 = (W_2, R_2)$
- $\mathcal{F}_1 \rhd \mathcal{F}_2 = (W, S_1, S_2)$ where
  - $W = W_1 \times W_2$
  - $(x_1, x_2) S_1 (y_1, y_2)$ iff $x_1 R_1 y_1$
  - $(x_1, x_2) S_2 (y_1, y_2)$ iff $x_1 = y_1$ and $x_2 R_2 y_2$

Proposition: Let $\mathcal{F} = (W, S_1, S_2)$ be countable, reflexive and such that

- $S_1 \circ S_2 \subseteq S_1$
- $S_2 \circ S_1 \subseteq S_1$
- $S_2^{-1} \circ S_1 \subseteq S_1$

There exists $\mathcal{F}_1 = (W_1, R_1)$, $\mathcal{F}_2 = (W_2, R_2)$ such that $\mathcal{F}$ is a bounded morphic image of $\mathcal{F}_1 \rhd \mathcal{F}_2$

Lexicographic product of normal modal logics

Lexicographic product of relational structures

- $\mathcal{F}_1 = (W_1, R_1)$, $\mathcal{F}_2 = (W_2, R_2)$
- $\mathcal{F}_1 \rhd \mathcal{F}_2 = (W, S_1, S_2)$ where
  - $W = W_1 \times W_2$
  - $(x_1, x_2)S_1(y_1, y_2)$ iff $x_1 R_1 y_1$
  - $(x_1, x_2)S_2(y_1, y_2)$ iff $x_1 = y_1$ and $x_2 R_2 y_2$

Combining normal modal logics: lexicographic product

- $L_1 \rhd L_2 = Log\{\mathcal{F}_1 \rhd \mathcal{F}_2 : \ \mathcal{F}_1 \models L_1$ and $\mathcal{F}_2 \models L_2\}$

Lexicographic product of normal modal logics

Product matching normal modal logics

- L_1 and L_2 are $\triangleright$-product matching iff $L_1 \triangleright L_2 =$
  $L_1 \otimes L_2 \otimes \Diamond_1\Diamond_2 p \to \Diamond_1 p \otimes \Diamond_2\Diamond_1 p \to \Diamond_1 p \otimes \Diamond_1 p \to \Box_2\Diamond_1 p$

Proposition: Let L_1 and L_2 be normal modal logics from the following list

- T, $S4$, $S5$

L_1 and L_2 are $\triangleright$-product matching

Limited hyperreals

Hyperreals: $(Hy, <_1, <_2)$

- Hy is the set of all limited hyperreals
- $x <_1 y$ iff $x < y$ and $y - x$ is not infinitesimal
- $x <_2 y$ iff $x < y$ and $y - x$ is infinitesimal

Proposition: $(Hy, <_1, <_2)$ is elementary equivalent to $(\mathbb{R}, <) \triangleright (\mathbb{R}, <)$

Limited hyperreals

Hyperreals: $(Hy, <_1, <_2)$

- Hy is the set of all limited hyperreals
- $x <_1 y$ iff $x < y$ and $y - x$ is not infinitesimal
- $x <_2 y$ iff $x < y$ and $y - x$ is infinitesimal

A first order theory: HY

- $\forall x\ x \not<_i x$
- $\forall x\ \exists y\ x <_i y$
- $\forall x\ \exists y\ y <_i x$
- $\forall x\ \forall y\ (\exists z\ (x <_i z \wedge z <_j y) \rightarrow x <_k y)$ where $k = \min\{i, j\}$
- $\forall x\ \forall y\ (x <_k y \rightarrow \exists z\ (x <_i z \wedge z <_j y))$ where $k = \min\{i, j\}$
- $\forall x\ \forall y (x = y \vee x <_1 y \vee x <_2 y \vee y <_1 x \vee y <_2 x)$

Limited hyperreals

Hyperreals: $(Hy, <_1, <_2)$

- Hy is the set of all limited hyperreals
- $x <_1 y$ iff $x < y$ and $y - x$ is not infinitesimal
- $x <_2 y$ iff $x < y$ and $y - x$ is infinitesimal

Proposition: The first-order theory HY is

- countably categorical
- not categorical in any uncountable power
- maximal consistent
- complete with respect to the lexicographic product of any dense linear orders without endpoints
- *PSPACE*-complete

Lexicographic product of linear temporal logics

Syntax

- $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid G_i\phi \mid H_i\phi$
- $F_i\phi ::= \neg G_i\neg\phi$, $F\phi ::= F_1\phi \vee F_2\phi$
- $P_i\phi ::= \neg H_i\neg\phi$, $P\phi ::= P_1\phi \vee P_2\phi$

Intuitive reading

- $G_1\phi$: "ϕ will be true at each instant within the future of but not infinitely close to the present instant"
- $G_2\phi$: "ϕ will be true at each instant within the future of and infinitely close to the present instant"
- $H_1\phi$: "ϕ has been true at each instant within the past of but not infinitely close to the present instant"
- $H_2\phi$: "ϕ has been true at each instant within the past of and infinitely close to the present instant"

Lexicographic product of linear temporal logics

Semantics

- Frames: lexicographic products $\mathcal{F}_1 \rhd \mathcal{F}_2 = (W, S_1, S_2)$ of two dense linear orders $\mathcal{F}_1 = (W_1, R_1)$, $\mathcal{F}_2 = (W_2, R_2)$ without endpoints
- Models: $\mathcal{M} = (W, S_1, S_2, V)$ where $V: \; p \mapsto \; V(p) \subseteq W$

Truth conditions

- $\mathcal{M}, (x_1, x_2) \models p$ iff $(x_1, x_2) \in V(p)$
- $\mathcal{M}, (x_1, x_2) \models G_i\phi$ iff for all $(y_1, y_2) \in W$, if $(x_1, x_2)S_i(y_1, y_2)$, $\mathcal{M}, (y_1, y_2) \models \phi$
- $\mathcal{M}, (x_1, x_2) \models H_i\phi$ iff for all $(y_1, y_2) \in W$, if $(y_1, y_2)S_i(x_1, x_2)$, $\mathcal{M}, (y_1, y_2) \models \phi$

Lexicographic product of linear temporal logics

Examples of valid formulas

- $F_i \top$
- $F_i F_j \phi \to F_k \phi$ where $k = \min\{i, j\}$
- $F_k \phi \to F_i F_j \phi$ where $k = \min\{i, j\}$
- $F_1 \phi \wedge F_1 \psi \to F_1(\phi \wedge \psi) \vee F_1(\phi \wedge F\psi) \vee F_1(\psi \wedge F\phi)$
- $F_1 \phi \wedge F_2 \psi \to F_2(\psi \wedge F_1 \phi)$
- $F_2 \phi \wedge F_1 \psi \to F_2(\phi \wedge F_1 \psi)$
- $F_2 \phi \wedge F_2 \psi \to F_2(\phi \wedge \psi) \vee F_2(\phi \wedge F_2 \psi) \vee F_2(\psi \wedge F_2 \phi)$

Axiomatization/completeness

Axiomatization: Let *HTL* be the least temporal logic in our language that contains the following formulas and their mirror images as proper axioms

- $F_i \top$
- $F_i F_j p \to F_k p$ where $k = \min\{i, j\}$
- $F_k p \to F_i F_j p$ where $k = \min\{i, j\}$
- $F_1 p \wedge F_1 q \to F_1(p \wedge q) \vee F_1(p \wedge Fq) \vee F_1(q \wedge Fp)$
- $F_1 p \wedge F_2 q \to F_2(q \wedge F_1 p)$
- $F_2 p \wedge F_1 q \to F_2(p \wedge F_1 q)$
- $F_2 p \wedge F_2 q \to F_2(p \wedge q) \vee F_2(p \wedge F_2 q) \vee F_2(q \wedge F_2 p)$

Axiomatization/completeness

Completeness:

- First, remark that all axioms are Sahlqvist formulas
- Second, by Sahlqvist completeness theorem and Löwenheim-Skolem theorem for modal models, infer that every consistent formula is satisfied in a countable model verifying the Sahlqvist conditions corresponding to the axioms (countable prestandard model)
- Third, prove that every countable prestandard model is a bounded morphic image of the lexicographic product of two dense linear orders without endpoints

Axiomatization/completeness

Pure future formulas: $\phi, \psi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid G_i\phi$

- There is no known complete axiomatization of the set of all valid pure future formulas
- The proper axioms of $HTL(G_1)$ are
  - $F_1\top$
  - $F_1F_1p \rightarrow F_1p$
  - $F_1p \rightarrow F_1F_1p$
  - $F_1(G_1p \wedge F_1q) \rightarrow G_1(p \vee F_1q)$
- The proper axioms of $HTL(G_2)$ are
  - $F_2\top$
  - $F_2F_2p \rightarrow F_2p$
  - $F_2p \rightarrow F_2F_2p$
  - $F_2p \wedge F_2q \rightarrow F_2(p \wedge q) \vee F_2(p \wedge F_2q) \vee F_2(q \wedge F_2p)$

Decidability/complexity

Proposition: The temporal logic *HTL* is

- *coNP*-complete

Proof: By the mosaic method

Let ξ be a fixed formula and Γ be the least set of formulas closed under subformulas and such that

- $\top \in \Gamma$
- $\Diamond_2 \xi \in \Gamma$
- Γ is closed under single negations
- if either $G_1\phi \in \Gamma$, or $H_1\phi \in \Gamma$, $G_2\phi \in \Gamma$ and $H_2\phi \in \Gamma$
- if either $F_1\phi \in \Gamma$, or $P_1\phi \in \Gamma$, $\Diamond_2\phi \in \Gamma$

Decidability/complexity

A function λ with *dom*$(\lambda) \subseteq \Gamma$ and *ran*$(\lambda) \subseteq \{0, 1\}$ is **adequate** iff

- *dom*(λ) is closed under single negations
- $\perp \in$ *dom*(λ) and $\lambda(\perp) = 0$
- if $\neg\phi, \phi \in$ *dom*(λ), $\lambda(\neg\phi) = 1 - \lambda(\phi)$
- if $\phi \vee \psi, \phi \in$ *dom*(λ), $\lambda(\phi \vee \psi) \geq \lambda(\phi)$
- if $\phi \vee \psi, \psi \in$ *dom*(λ), $\lambda(\phi \vee \psi) \geq \lambda(\psi)$
- if $\phi \vee \psi, \phi, \psi \in$ *dom*(λ), $\lambda(\phi \vee \psi) = \max\{\lambda(\phi), \lambda(\psi)\}$

Decidability/complexity

A pair (σ, τ) of adequate functions is **G_i-coherent** iff

- if $\sigma(G_i\phi) = 1$, $\tau(G_i\phi) = 1$ and $\tau(\phi) = 1$

A pair (σ, τ) of adequate functions is **H_i-coherent** iff

- if $\tau(H_i\phi) = 1$, $\sigma(H_i\phi) = 1$ and $\sigma(\phi) = 1$

A **1-mosaic** is a $\{G_1, H_1\}$-coherent pair (σ, τ) of adequate functions such that

- if $\sigma(G_1\phi) = 1$, $\tau(G_2\phi) = 1$ and $\tau(H_2\phi) = 1$
- if $\tau(H_1\phi) = 1$, $\sigma(G_2\phi) = 1$ and $\sigma(H_2\phi) = 1$

A **2-mosaic** is a $\{G_2, H_2\}$-coherent pair (σ, τ) of adequate functions with domain Γ and such that

- $\sigma(G_1\phi) = \tau(G_1\phi)$
- $\sigma(H_1\phi) = \tau(H_1\phi)$

Decidability/complexity

An **1-saturated set of mosaics** (1-SSM) is a collection M of 1-mosaics such that

- if $(\sigma, \tau) \in M$, there is μ such that $(\tau, \mu) \in M$
- if $(\sigma, \tau) \in M$, there is μ such that $(\mu, \sigma) \in M$
- if $(\sigma, \tau) \in M$, there is μ such that $(\sigma, \mu), (\mu, \tau) \in M$
- if $(\sigma, \tau) \in M$, $\sigma(F_1\phi) = 1$ and $\tau(F_1\phi) = \tau(\Diamond_2\phi) = 0$, there is μ such that $\mu(\Diamond_2\phi) = 1$ and $(\sigma, \mu), (\mu, \tau) \in M$
- if $(\sigma, \tau) \in M$, $\tau(P_1\phi) = 1$ and $\sigma(P_1\phi) = \sigma(\Diamond_2\phi) = 0$, there is μ such that $\mu(\Diamond_2\phi) = 1$ and $(\sigma, \mu), (\mu, \tau) \in M$
- if $(\sigma, \tau) \in M$ and $\tau(F_1\phi) = 1$, there is μ such that $\mu(\Diamond_2\phi) = 1$ and $(\tau, \mu) \in M$
- if $(\sigma, \tau) \in M$ and $\sigma(P_1\phi) = 1$, there is μ such that $\mu(\Diamond_2\phi) = 1$ and $(\mu, \sigma) \in M$

Decidability/complexity

An **2-saturated set of mosaics** (2-SSM) is a collection M of 2-mosaics such that

- if $(\sigma, \tau) \in M$, there is μ such that $(\tau, \mu) \in M$
- if $(\sigma, \tau) \in M$, there is μ such that $(\mu, \sigma) \in M$
- if $(\sigma, \tau) \in M$, there is μ such that $(\sigma, \mu), (\mu, \tau) \in M$
- if $(\sigma, \tau) \in M$, $\sigma(F_2\phi) = 1$ and $\tau(F_2\phi) = \tau(\phi) = 0$, there is μ such that $\mu(\phi) = 1$ and $(\sigma, \mu), (\mu, \tau) \in M$
- if $(\sigma, \tau) \in M$, $\tau(P_2\phi) = 1$ and $\sigma(P_2\phi) = \sigma(\phi) = 0$, there is μ such that $\mu(\phi) = 1$ and $(\sigma, \mu), (\mu, \tau) \in M$
- if $(\sigma, \tau) \in M$ and $\tau(F_2\phi) = 1$, there is μ such that $\mu(\phi) = 1$ and $(\tau, \mu) \in M$
- if $(\sigma, \tau) \in M$ and $\sigma(P_2\phi) = 1$, there is μ such that $\mu(\phi) = 1$ and $(\mu, \sigma) \in M$

Decidability/complexity

Given a 2-SSM M, we define a function λ_M by

- $\lambda_M(\phi) = 0$ if for all $(\sigma, \tau) \in M$, $\sigma(\phi) = \tau(\phi) = 0$
- $\lambda_M(\phi) = 1$ if for all $(\sigma, \tau) \in M$, $\sigma(\phi) = \tau(\phi) = 1$
- $\lambda_M(\phi)$ is undefined otherwise

A **1-supermosaic** is a pair (M, N) of 2-SSM such that

- (λ_M, λ_N) is a 1-mosaic

A **saturated set of 1-supermosaics** (1-SSS) is a collection Σ of 1-supermosaics such that

- $\{(\lambda_M, \lambda_N) : (M, N) \in \Sigma\}$ is a 1-SSM

Decidability/complexity

A **1-SSS for ξ** is a 1-SSS Σ such that

- there is $(M, N) \in \Sigma$ such that either $\lambda_M(\Diamond_2\xi) = 1$, or $\lambda_N(\Diamond_2\xi) = 1$

Proposition: If there exists a 1-SSS for ξ, ξ is satisfiable in $(\mathbb{Q}, <) \triangleright (\mathbb{Q}, <)$

Proposition: If ξ is satisfiable in $(\mathbb{Q}, <) \triangleright (\mathbb{Q}, <)$, there exists a 1-SSS for ξ of cardinal bounded by $(4 \times \mathit{Card}(\Gamma))^4$

Proposition: The temporal logic *HTL* is

- *coNP*-complete

Conclusion and open problems

Conclusion

- Lexicographic products of unbounded dense linear orders
- First-order theory: *HY*
- Temporal logic: *HTL*
- Axiomatization/completeness
- Decidability/complexity

Conclusion and open problems

Open problems

1. The modal logics characterized by the lexicographic products of two given linear orderings among $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are all equal to $S4.3 \triangleright S4.3$. HTL is the temporal logic characterized by $(\mathbb{Q}, <) \triangleright (\mathbb{Q}, <)$. What are the temporal logics characterized by, for example, $(\mathbb{Z}, <) \triangleright (\mathbb{Z}, <)$ and $(\mathbb{R}, <) \triangleright (\mathbb{R}, <)$?

Conclusion and open problems

Open problems

2. The formation of fusion has nice features, seeing that, for instance, Kripke completeness and decidability of modal logics L_1 and L_2 are transferred to their fusion $L_1 \otimes L_2$. Could transfer results similar to the ones obtained by Kracht and Wolter (1991) be obtained in our lexicographic setting?

Conclusion and open problems

Open problems

3. All extensions of $S4.3$, as proved by Bull (1966), possess the finite model property. All finitely axiomatizable normal extensions of $K4.3$, as proved by Zakharyaschev and Alekseev (1995), are decidable. Is it possible to obtain similar results in our lexicographic setting? Or could undecidability results similar to the ones obtained by Reynolds and Zakharyaschev (2001) within the context of the asynchronous products of the modal logics determined by arbitrarily long linear orders be obtained in our lexicographic setting?

Conclusion and open problems

Open problems

4. There is also the question of associating with $<_1$ and $<_2$ the until-like connectives U_1 and U_2 and the since-like connectives S_1 and S_2, the formulas $\phi U_1 \psi$, $\phi U_2 \psi$, $\phi S_1 \psi$ and $\phi S_2 \psi$ being read as one reads the formulas $\phi U \psi$ and $\phi S \psi$ in classical temporal logic, this time with $<_1$ and $<_2$. As yet, nothing has been done concerning the issues of axiomatization/completeness and decidability/complexity these new temporal connectives give rise to.